How many solutions?

Hung-yi Lee

Review

Given a system of linear equations with m equations and n variables

$A\mathbf{x} = \mathbf{b}$ $\quad A: m \times n \quad x \in R^n \quad b \in R^m$

Is b a linear combination of columns of A?

Is b in the span of the columns of A?

NO

No solution

YES

Have solution

We don't know how many solutions

Today

Given a system of linear equations with m equations and n variables

$\mathbf{A}\mathbf{x} = \mathbf{b}$ $\quad A: m \times n$ $\quad x \in R^n$ $\quad b \in R^m$

Is b a linear combination of columns of A?

Is b in the span of the columns of A?

NO

No solution

Other cases?

YES

The columns of A are ***independent***.

Rank A = n

Nullity A = 0

Unique solution

The columns of A are ***dependent***.

Rank A < n

Nullity A > 0

Infinite solution

How many solutions?

Dependent and Independent

Dependent and Independent

Linear Dependent

Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, if there exists any $\mathbf{a}_i$ that is a linear combination of other vectors

$$\left\{\begin{bmatrix}-4\\12\\6\end{bmatrix}, \begin{bmatrix}-10\\30\\15\end{bmatrix}\right\}$$

Dependent or Independent?

$$\left\{\begin{bmatrix}6\\3\\3\end{bmatrix}, \begin{bmatrix}1\\8\\3\end{bmatrix}, \begin{bmatrix}7\\11\\6\end{bmatrix}\right\}$$

Dependent or Independent?

Dependent and Independent

Linear Dependent

Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, if there exists any $\mathbf{a}_i$ that is a linear combination of other vectors

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix} \right\}$$

Dependent or Independent?

Zero vector is the linear combination of any other vectors

Any set contains zero vector would be linear dependent

Flaw of the definition: How about a set with only one vector?

How to check?

Linear Dependent

Given a vector set, {$\mathbf{a}_1$, $\mathbf{a}_2$, …, $\mathbf{a}_n$}, if there exists any $\mathbf{a}_i$ that is a linear combination of other vectors

$$2\boldsymbol{a_i} + \boldsymbol{a_j} + 3\boldsymbol{a_k} = \mathbf{0}$$

$$2\boldsymbol{a_i} + \boldsymbol{a_j} = -3\boldsymbol{a_k}$$

$$\left(-\frac{2}{3}\right)\boldsymbol{a_i} + \left(-\frac{1}{3}\right)\boldsymbol{a_j} = \boldsymbol{a_k}$$

$\Uparrow \Downarrow$

$$\boldsymbol{a_{i'}} = 3\boldsymbol{a_{j'}} + 4\boldsymbol{a_{k'}}$$

$$\boldsymbol{a_{i'}} - 3\boldsymbol{a_{j'}} - 4\boldsymbol{a_{k'}} = \mathbf{0}$$

Given a vector set, {$\mathbf{a}_1$, $\mathbf{a}_2$, …, $\mathbf{a}_n$}, there exists scalars x_1, x_2, …, x_n, that are **not all zero**, such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$.

Another Definition

How about the vector with only one element?

- A set of n vectors $\{\boldsymbol{a_1}, \boldsymbol{a_2}, \cdots, \boldsymbol{a_n}\}$ is linear dependent
  - If there exist scalars $x_1, x_2, \cdots, x_n$, **not all zero**, such that

$$x_1\boldsymbol{a_1} + x_2\boldsymbol{a_2} + \cdots + x_n\boldsymbol{a_n} = \boldsymbol{0}$$

- A set of n vectors $\{\boldsymbol{a_1}, \boldsymbol{a_2}, \cdots, \boldsymbol{a_n}\}$ is linear independent
  - Only scalars such that

$$x_1\boldsymbol{a_1} + x_2\boldsymbol{a_2} + \cdots + x_n\boldsymbol{a_n} = \boldsymbol{0}$$

Only if $x_1 = x_2 = \cdots = x_k = 0$

Intuition

Dependent:
Once we have solution, we have infinite.

- Intuitive link between dependence and the number of solutions

$$\begin{bmatrix} 6 & 1 & 7 \\ 3 & 8 & 11 \\ 3 & 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 22 \\ 12 \end{bmatrix} \qquad 1 \cdot \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix}$$

$$1 \cdot \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix} = \begin{bmatrix} 14 \\ 22 \\ 12 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$2 \cdot \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} + 2 \cdot \begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 22 \\ 12 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

Infinite Solution

Homogeneous Equations

$$x_1\boldsymbol{a_1} + x_2\boldsymbol{a_2} + \cdots + x_n\boldsymbol{a_n} = \mathbf{0} \iff Ax = \mathbf{0}$$

$$A = [a_1 \quad a_2 \quad \cdots \quad a_n]$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Homogeneous linear equations

Always having $\mathbf{0}$ as solution

A set of n vectors $\{\boldsymbol{a_1}, \boldsymbol{a_2}, \cdots, \boldsymbol{a_n}\}$ is linear dependent $\iff$ $Ax = \mathbf{0}$ have non-zero solution

infinite

A set of n vectors $\{\boldsymbol{a_1}, \boldsymbol{a_2}, \cdots, \boldsymbol{a_n}\}$ is linear independent $\iff$ $Ax = \mathbf{0}$ only have zero solution

Homogeneous Equations

- Columns of A are **dependent** → If Ax=b have solution, it will have Infinite Solutions

We can find non-zero solution u such that $Au = \mathbf{0}$

There exists v such that $Av = \mathrm{b}$

$A(u+v) = \mathrm{b}$

$u + v$ is another solution different to v

- If Ax=b have Infinite solutions → Columns of A are dependent

$Au = \mathrm{b}$

$Av = \mathrm{b}$

$$A\underbrace{(u-v)}_{\text{Non-zero}} = \mathbf{0}$$

How many solutions?

Rank and Nullity

Intuitive Definition

- The **rank** of a matrix is defined as the maximum number of *linearly independent columns* in the matrix.
- **Nullity** = Number of columns - **rank**

$$\begin{bmatrix} -3 & 2 & -1 \\ 7 & 9 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 10 \\ 2 & 6 & 20 \\ 3 & 9 & 30 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Intuitive Definition

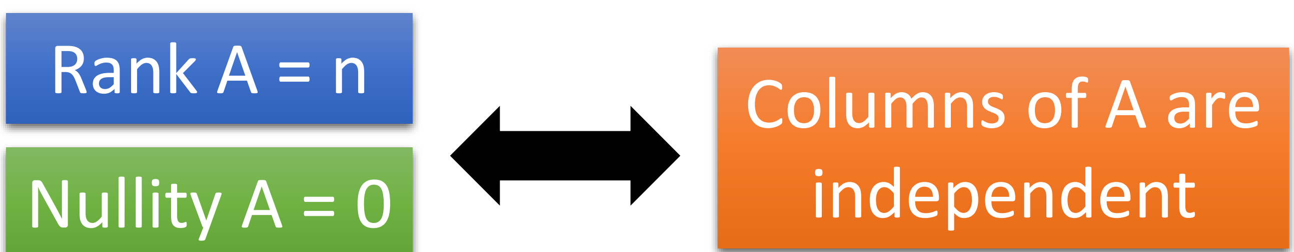
- The **rank** of a matrix is defined as the maximum number of *linearly independent columns* in the matrix.
- **Nullity** = Number of columns - **rank**

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{bmatrix} \qquad \begin{bmatrix} 0 & 3 \\ 0 & 5 \end{bmatrix} \qquad \begin{bmatrix} 5 \\ 2 \end{bmatrix} \qquad [6]$$

Intuitive Definition

- The **rank** of a matrix is defined as the maximum number of *linearly independent columns* in the matrix.
- **Nullity** = Number of columns - **rank**

If A is a mxn matrix:



How many solutions?

Concluding Remarks

Conclusion

$$A\mathbf{x} = \mathbf{b}$$

$A: m \times n \quad x \in R^n \quad b \in R^m$

Is b a linear combination of columns of A?

Is b in the span of the columns of A?

NO

No solution

YES

The columns of A are ***independent***.

Rank A = n

Nullity A = 0

Unique solution

The columns of A are ***dependent***.

Rank A < n

Nullity A > 0

Infinite solution

Conclusion

The columns of A are ***independent***.

Rank A = n

Nullity A = 0

$A: m \times n$

$x \in R^n$ $b \in R^m$

NO

Is b a linear combination of columns of A?

Is b in the span of the columns of A?

NO

No solution

YES

Infinite solution

YES

Is b a linear combination of columns of A?

Is b in the span of the columns of A?

NO

No solution

YES

Unique solution

Question

- True or False
  - If the columns of A are linear independent, then Ax=b has unique solution.
  - If the columns of A are linear independent, then Ax=b has at most one solution.
  - If the columns of A are linear dependent, then Ax=b has infinite solution.
  - If the columns of A are linear independent, then Ax=0 (homogeneous equation) has unique solution.
  - If the columns of A are linear dependent, then Ax=0 (homogeneous equation) has infinite solution.

Acknowledgement

- 感謝 姚鈞嚴 同學發現投影片 上的錯誤