Subspaces associated with a Matrix

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Reference

- Textbook: Chapter 4.3

Three Associated Subspaces

- A is an m x n matrix

Col A

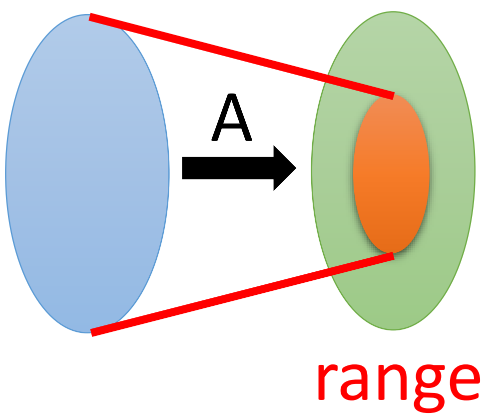
in R^m

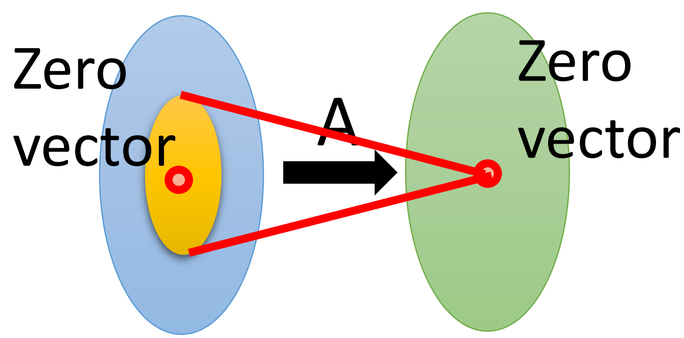
Null A

in R^n

Row A

in R^n





= Col A^T

Basis?

Dimension?

Col A

- Basis: The pivot columns of A form a basis for Col A.

$$A = \begin{bmatrix} \boxed{\begin{matrix} 1 \\ -1 \\ 2 \\ -3 \end{matrix}} & \begin{matrix} 2 \\ -2 \\ 4 \\ -6 \end{matrix} & \boxed{\begin{matrix} -1 \\ 1 \\ -3 \\ 2 \end{matrix}} & \boxed{\begin{matrix} 2 \\ 2 \\ 2 \\ 0 \end{matrix}} & \begin{matrix} 1 \\ 3 \\ 0 \\ 3 \end{matrix} & \begin{matrix} 2 \\ 6 \\ 3 \\ 9 \end{matrix} \end{bmatrix} \Rightarrow \text{Col } A = \text{Span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix} \right\}$$

pivot columns

pivot columns

- Dimension:

Dim (Col A) = number of pivot columns

= rank A

Rank A (revisit)

- Maximum number of Independent Columns
- Number of Pivot Columns
- Number of Non-zero rows
- Number of Basic Variables
- Dim (Col A): dimension of column space
- Dimension of the range of A

Example 2, P256

Null A

$$A = \begin{bmatrix} 3 & 1 & -2 & 1 & 5 \\ 1 & 0 & 1 & 0 & 1 \\ -5 & -2 & 5 & -5 & -3 \\ -2 & -1 & 3 & 2 & -10 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -5 & 0 & 4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Basis:
  - Solving Ax = 0
  - Each free variable in the parametric representation of the general solution is multiplied by a vector.
  - The vectors form the basis.

$$\begin{aligned} x_1 + x_3 + x_5 &= 0 \\ x_2 - 5x_3 + 4x_5 &= 0 \\ x_4 - 2x_5 &= 0 \end{aligned} \Rightarrow \begin{aligned} x_1 &= -x_3 - x_5 \\ x_2 &= 5x_3 - 4x_5 \\ x_3 &= x_3 \text{ (free)} \\ x_4 &= 2x_5 \\ x_5 &= x_5 \text{ (free)} \end{aligned} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -4 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Basis

Null A

- Basis:
  - Solving $Ax = 0$
  - Each free variable in the parametric representation of the general solution is multiplied by a vector.
  - The vectors form the basis.
- Dimension:

$$\begin{aligned} \text{Dim (Null A)} &= \text{number of free variables} \\ &= \text{Nullity A} \\ &= n - \text{Rank A} \end{aligned}$$

Row A

- Basis: Nonzero rows of RREF(A)

$$A = \begin{bmatrix} 3 & 1 & -2 & 1 & 5 \\ 1 & 0 & 1 & 0 & 1 \\ -5 & -2 & 5 & -5 & -3 \\ -2 & -1 & 3 & 2 & -10 \end{bmatrix} \xrightarrow{\text{RREF}} R = \begin{bmatrix} \boxed{1 \quad 0 \quad 1 \quad 0 \quad 1} \\ \boxed{0 \quad 1 \quad -5 \quad 0 \quad 4} \\ \boxed{0 \quad 0 \quad 0 \quad 1 \quad -2} \\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \end{bmatrix}$$

Row A = Row R

(The elementary row operations do not change the row space.)

a basis of Row *R*

= a basis of Row *A*

- Dimension: Dim (Row A) = Number of Nonzero rows

  = Rank A

Rank A (revisit)

- Maximum number of Independent Columns
- Number of Pivot Column
- Number of Non-zero rows
- Number of Basic Variables
- Dim (Col A): dimension of column space = Dim (Row A)
- Dimension of the range of A = Dim (Col A^T)

Rank A = Rank A^T

- Proof

Rank A = Dim (Col A)

Rank A = Dim (Row A)

= Dim (Col A^T)

= Rank A^T

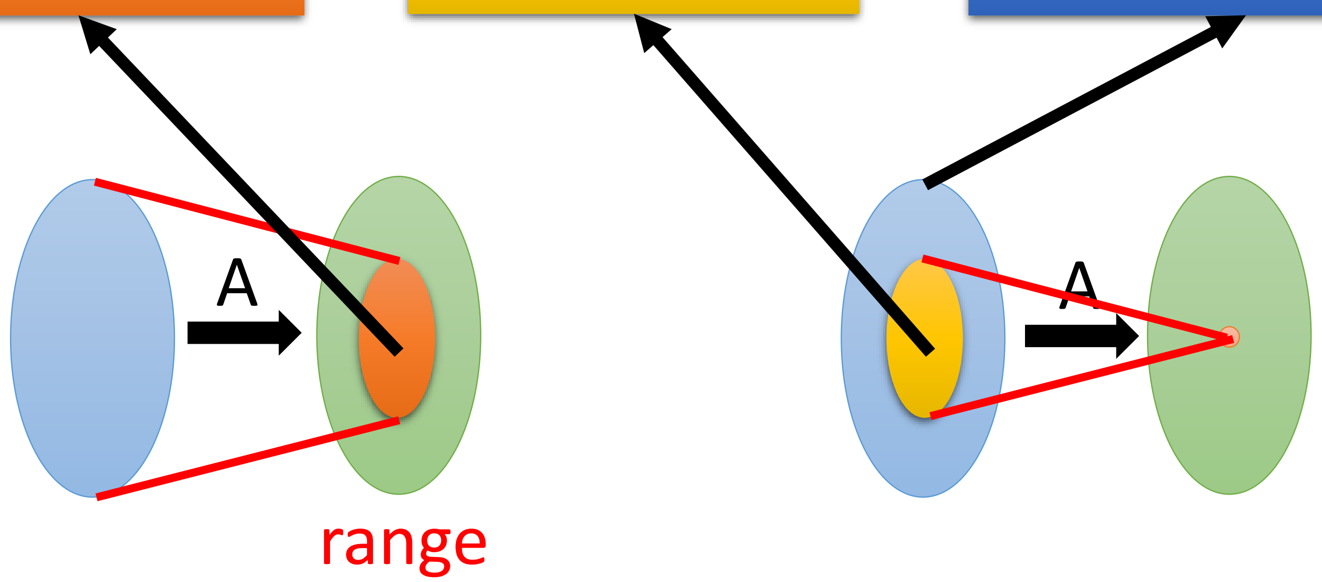
Dimension Theorem

Dim (Col A) =Rank A

Dim (Null A) =n - Rank A

If A is mxn Dim (R^n) =n

Dim of Range + Dim of Null = Dim of Domain



Summary

A is an m x n matrix

	Dimension	Basis
Col A	Rank A	The pivot columns of A
Null A	Nullity A = n - Rank A	The vectors in the parametric representation of the solution of Ax=0
Row A	Rank A	The nonzero rows of the RREF of A