Determinant
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期中考

- 範圍：ch 1 ~ ch 4
- 時間：11/09 (五) 上課時間
- 地點：會公告在 ceiba 上
Reference

• MIT OCW Linear Algebra:
  • Lecture 18: Properties of determinants
  • Lecture 19: Determinant formulas and cofactors
  • Lecture 20: Cramer's rule, inverse matrix, and volume

• Textbook: Chapter 3
Determinant

- The determinant of a **square matrix** is a **scalar** that provides information about the matrix.
  - E.g. **Invertibility** of the matrix.

- Learning Target
  - The formula of Determinants
  - The properties of Determinants
  - Cramer’s Rule
Formula for Determinants
Determinants in High School

• 2 X 2

\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]

\[ \text{det}(A) = ad - bc \]

• 3 X 3

\[ A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \]

\[ \text{det}(A) = a_1a_5a_9 + a_2a_6a_7 + a_3a_4a_8 \]
\[ -a_3a_5a_7 - a_2a_4a_9 - a_1a_6a_8 \]
Cofactor Expansion

• Suppose $A$ is an $n \times n$ matrix. $A_{ij}$ is defined as the submatrix of $A$ obtained by removing the $i$-th row and the $j$-th column.
Cofactor Expansion

• Pick row 1
  \[ \det A = a_{11}c_{11} + a_{12}c_{12} + \cdots + a_{1n}c_{1n} \]

• Or pick row \( i \)
  \[ \det A = a_{i1}c_{i1} + a_{i2}c_{i2} + \cdots + a_{in}c_{in} \]

• Or pick column \( j \)
  \[ \det A = a_{1j}c_{1j} + a_{2j}c_{2j} + \cdots + a_{nj}c_{nj} \]

\[ c_{ij} = (-1)^{i+j} \det A_{ij} \]

Cofactor expansion again......
2 x 2 matrix

- Define \( \text{det}([a]) = a \)

\[
A = \begin{bmatrix}
 a & b \\
 c & d \\
\end{bmatrix}
\]

Pick the first row

\[
\text{det}(A) = ac_{11} + bc_{12}
\]

\[
c_{11} = (-1)^{1+1} \text{det}([d]) = d
\]

\[
c_{12} = (-1)^{1+2} \text{det}([c]) = -c
\]

\[
d \text{det}(A) = ad - bc
\]
3 x 3 matrix

\[ A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \]

\[ c_{ij} = (-1)^{i+j} \det A_{ij} \]

\[ \det A = a_{21}c_{21} + a_{22}c_{22} + a_{23}c_{23} \]

\[ \begin{align*}
(-1)^{2+1} \det A_{21} & = (-1) \cdot 4 \cdot 5 \cdot 6 \\
(-1)^{2+2} \det A_{22} & = (-1) \cdot 4 \cdot 5 \cdot 6 \\
(-1)^{2+3} \det A_{23} & = (-1) \cdot 4 \cdot 5 \cdot 6 
\end{align*} \]

\[ A_{21} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad A_{22} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad A_{23} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \]
Example

- Given tridiagonal $n \times n$ matrix $A$

\[
A = \begin{bmatrix}
1 & 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\
1 & 1 & 1 & \cdots & \cdots & 0 & 0 & 0 \\
0 & 1 & 1 & \cdots & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \cdots & 1 & 1 & 0 \\
0 & 0 & 0 & \cdots & \cdots & 1 & 1 & 1 \\
0 & 0 & 0 & \cdots & \cdots & 0 & 1 & 1
\end{bmatrix}
\]

Find $\det A$ when $n = 999$
\[ \det A_4 = a_{11} c_{11} + a_{12} c_{12} + a_{13} c_{13} + a_{14} c_{14} \]

\[ A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \]

\[ A_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \]

\[ A_4 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \]

\[ c_{11} = (-1)^2 \det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \det(A_3) \]

\[ c_{12} = (-1)^3 \det \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = -\det(A_2) \]

\[ = \det(A_2) \]
Example

\[ \det(A_4) = \det(A_3) - \det(A_2) \]

\[ \det(A_n) = \det(A_{n-1}) - \det(A_{n-2}) \]

\[ \det(A_1) = 1 \quad \det(A_2) = 0 \quad \det(A_3) = -1 \]

\[ \det(A_4) = -1 \quad \det(A_5) = 0 \quad \det(A_6) = 1 \]

\[ \det(A_7) = 1 \quad \det(A_8) = 0 \quad \ldots \ldots \]
Properties of Determinants

“Volume” in high dimensions (?)
Determinants in High School

• 2 × 2

\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]

• 3 × 3

\[ A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \]
Three Basic Properties

• Basic Property 1: $\det(I) = 1$
• Basic Property 2: Exchange rows only reverses the sign of det (do not change absolute value)
• Basic Property 3: Determinant is “linear” for each row

Area in 2d and Volume in 3d have the above properties

Can we say determinant is the “Volume” also in high dimension?
Three Basic Properties

• Basic Property 1:
  • $det(I) = 1$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$det(I_2) = 1$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$det(I_3) = 1$$

正方形 面積為 1

正立方體 體積為 1
Three Basic Properties

- Basic Property 2:
  - Exchanging rows only reverses the sign of \( \det \)

\[
\begin{align*}
\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} &= 1 \\
\det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &= -1
\end{align*}
\]

\[
\begin{align*}
\det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} &= 1 \\
\det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} &= -1 \\
\det \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} &= 1
\end{align*}
\]
Three Basic Properties

• Basic Property 2:
  • Exchanging rows only reverses the sign of det

If a matrix $A$ has 2 equal rows

$$\det(A) = 0$$

exchanging two rows

$$A \xrightarrow{\text{exchange two rows}} A'$$

$$\det(A) = K \quad = \quad \det(A') = -K$$

Exchanging the two equal rows yields the same matrix
Three Basic Properties

• Basic Property 3:
  • Determinant is “linear” for each row

\[ \text{det} \left( \begin{bmatrix} ta & tb \\ c & d \end{bmatrix} \right) = t \text{det} \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \]
Three Basic Properties

• Basic Property 3:
  • Determinant is “linear” for each row

\[ \text{det} \begin{bmatrix} ta & t \ b \\ c & d \end{bmatrix} = t \text{det} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]

Q: find \( \text{det}(2A) \)

If \( A \) is \( n \times n \) ......

A: \( \text{det}(2A) = 2^n \text{det}(A) \)
Three Basic Properties

• Basic Property 3:
  • Determinant is “linear” for each row

3-a
\[
\det \left( \begin{bmatrix} ta & tb \\ c & d \end{bmatrix} \right) = t \det \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)
\]

A row of zeros  \( \rightarrow \)  \( \det(A) = 0 \)
Set \( t = 0 \)!

A row of zeros  \( \rightarrow \)  “volume” is zero
Three Basic Properties

• Basic Property 3:
  • Determinant is “linear” for each row

\[
3-b \quad \text{det} \left( \begin{bmatrix} a + a' & b + b' \\ c & d \end{bmatrix} \right) = \text{det} \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) + \text{det} \left( \begin{bmatrix} a' & b' \\ c & d \end{bmatrix} \right)
\]
Three Basic Properties

• Basic Property 3:
  • Determinant is "linear" for each row

Subtract $k \times$ row $i$ from row $j$ (elementary row operation)

$\text{det}\left(\begin{bmatrix} a & b \\ c - ka & d - kb \end{bmatrix}\right)$

3-b = $\text{det}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + \text{det}\left(\begin{bmatrix} a & b \\ -ka & -kb \end{bmatrix}\right)$

3-a = $\text{det}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) - k\text{det}\left(\begin{bmatrix} a & b \\ a & b \end{bmatrix}\right) = \text{det}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$
Determinants for Upper Triangular Matrix

\[ U = \begin{bmatrix} d_1 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix} \]

Killing everything above

Does not change the det

\[ \det(U) = \det \left( \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix} \right) \]

Property 1

3-a = d_1 d_2 \cdots d_n \det \left( \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \right) = 1

\[ \det(U) = d_1 d_2 \cdots d_n \text{ (Products of diagonal)} \]
Determinants v.s. Invertible

- **A is invertible** ↔ **det(A) ≠ 0**

**A** → **R**

**Elementary row operation**

- **det(A)**
- **det(R)**

\[
\text{det}(R) = \pm k_1 k_2 \cdots \text{det}(A)
\]

**Exchange**: Change sign

- If A is invertible, R is identity
  \[
  \text{det}(R) = 1 \quad \text{⇒} \quad \text{det}(A) \neq 0
  \]

**Scaling**: Multiply k

- If A is not invertible, R has zero row
  \[
  \text{det}(R) = 0 \quad \text{⇒} \quad \text{det}(A) = 0
  \]

**Add row**: nothing
### Invertible

- Let $A$ be an $n \times n$ matrix. $A$ is invertible if and only if:
  - The columns of $A$ span $\mathbb{R}^n$
  - For every $b$ in $\mathbb{R}^n$, the system $Ax = b$ is consistent
  - The rank of $A$ is $n$
  - The columns of $A$ are linear independent
  - The only solution to $Ax = 0$ is the zero vector
  - The nullity of $A$ is zero
  - The reduced row echelon form of $A$ is $I_n$
  - $A$ is a product of elementary matrices
  - There exists an $n \times n$ matrix $B$ such that $BA = I_n$
  - There exists an $n \times n$ matrix $C$ such that $AC = I_n$
  - $\det(A) \neq 0$
Example

A is invertible $\iff$ $\det(A) \neq 0$

$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 0 & c \\ 2 & 1 & 7 \end{bmatrix}$

For what scalar $c$ is the matrix not invertible?

$\det(A) = 0$

\[
\begin{align*}
detA &= 1 \cdot 0 \cdot 7 + (-1) \cdot c \cdot 2 + 2 \cdot (-1) \cdot 1 \\
&\quad -2 \cdot 0 \cdot 2 - (-1) \cdot (-1) \cdot 7 -1 \cdot c \cdot 1 \\
&= 0 - 2c - 2 - 7 - c = -3c - 9
\end{align*}
\]

not invertible $\iff$ $-3c - 9 = 0$ $\iff$ $c = -3$
More Properties of Determinants

• $\det(AB) = \det(A)\det(B)$
  
  Q: find $\det(A^{-1})$

  $\therefore A^{-1}A = I$  $\therefore \det(A^{-1})\det(A) = \det(I) = 1$

  $\therefore \det(A^{-1}) = 1/\det(A)$

  Q: find $\det(A^2)$

  $\det(A^2) = \det(A)\det(A) = \det(A)^2$

• $\det(A^T) = \det(A)$

  • Zero row $\rightarrow$ zero column
  
  • Same row $\rightarrow$ same column ......
Cramer’s Rule
Formula for $A^{-1}$

- $A^{-1} = \frac{1}{\text{det}(A)} C^T$
  - $\text{det}(A)$: scalar
  - $C$: cofactors of $A$ ($C$ has the same size as $A$, so does $C^T$)
  - $C^T$ is adjugate of $A$ (adj $A$, 伴隨矩陣)

\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}
\]
\[
\text{det}(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc
\]
\[
C^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\]
\[
A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\]
Formula for $A^{-1}$

$$A^{-1} = \frac{1}{\det(A)} C^T$$

- $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, A^{-1} = ?$

$$\det(A) = ae i + bf g + cd h - ce g - bd i - af h$$

$$C = \begin{bmatrix} + \begin{vmatrix} e & f \\ h & i \end{vmatrix} & - \begin{vmatrix} d & f \\ g & i \end{vmatrix} & + \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ - \begin{vmatrix} b & c \\ h & i \end{vmatrix} & + \begin{vmatrix} a & c \\ g & i \end{vmatrix} & - \begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ + \begin{vmatrix} b & c \\ e & f \end{vmatrix} & - \begin{vmatrix} a & c \\ d & f \end{vmatrix} & + \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}$$
Formula for $A^{-1}$

$$A^{-1} = \frac{1}{\det(A)} C^T$$

- Proof: $AC^T = det(A)I_n$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} c_{11} & \cdots & c_{n1} \\ \vdots & \ddots & \vdots \\ c_{1n} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} det(A) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & det(A) \end{bmatrix}$$

**Diagonal:** By definition of determinants

**Not Diagonal:**

(Exercise 82, P221)
Cramer’s Rule

\[ \begin{align*}
Ax &= b \\
x &= A^{-1}b \\
&= \frac{1}{\det(A)} C^T b
\end{align*} \]

\[ x_1 = \frac{\det(B_1)}{\det(A)} \]

\[ B_1 = \text{with column 1 replaced by } b \]

\[ B_1 = \begin{bmatrix} b \\ \vdots \\ \text{n-1 Columns of } A \\ B_j \end{bmatrix} \quad \text{with column } j \text{ replaced by } b \]
Appendix
Formula from Three Properties

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} = 1 \\
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} = -1
\]

\[
\det\begin{bmatrix}
a & b \\
c & d
\end{bmatrix} = \det\begin{bmatrix}
a & 0 \\
c & d
\end{bmatrix} + \det\begin{bmatrix}
0 & b \\
c & d
\end{bmatrix}
\]

\[
= \det\begin{bmatrix}
a & 0 \\
c & 0
\end{bmatrix} + \det\begin{bmatrix}
a & 0 \\
0 & d
\end{bmatrix} + \det\begin{bmatrix}
0 & b \\
c & 0
\end{bmatrix} + \det\begin{bmatrix}
0 & b \\
0 & d
\end{bmatrix}
\]

\[
= 0 + ad + (-bc) + 0 = ad - bc
\]

= \text{ad} - bc
Finally, we get 3 x 3 x 3 matrices

Most of them have zero determinants
det \[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

3! matrices have non-zero rows

\[
\begin{bmatrix}
  a_{11} & 0 & 0 \\
  0 & a_{22} & 0 \\
  0 & 0 & a_{33}
\end{bmatrix}
\]
\[
\begin{bmatrix}
  a_{11} & 0 & 0 \\
  0 & 0 & a_{23} \\
  0 & a_{32} & 0
\end{bmatrix}
\]
\[
\begin{bmatrix}
  0 & a_{12} & 0 \\
  a_{21} & 0 & 0 \\
  a_{31} & 0 & 0
\end{bmatrix}
\]
\[
\begin{bmatrix}
  a_{12} & 0 & 0 \\
  0 & a_{23} & 0 \\
  0 & 0 & a_{31}
\end{bmatrix}
\]
\[
\begin{bmatrix}
  a_{13} & 0 & 0 \\
  a_{21} & 0 & 0 \\
  a_{31} & 0 & 0
\end{bmatrix}
\]
\[
\begin{bmatrix}
  0 & 0 & a_{13} \\
  0 & a_{22} & 0 \\
  a_{31} & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
  a_{11} a_{22} a_{33} \\
  -a_{11} a_{23} a_{32} \\
  -a_{12} a_{21} a_{33}
\end{bmatrix}
\]
\[
\begin{bmatrix}
  a_{12} a_{23} a_{31} \\
  a_{13} a_{21} a_{32} \\
  -a_{13} a_{22} a_{31}
\end{bmatrix}
\]

Pick an element at each row, but they can not be in the same column.
Formula from Three Properties

• Given an n x n matrix A

\[ \det(A) = \sum n! \text{ terms} \]

Format of each term: \( a_{1\alpha}a_{2\beta}a_{3\gamma} \ldots a_{n\omega} \)

Find an element in each row

permutation of 1,2, ..., n
Example

\[
det \begin{pmatrix}
0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}
\]

\[
det \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\] + \[
det \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]
Formulas for Determinants

\[ detA = \sum \text{n! terms} \]

Format of each term: \( a_{1\alpha} a_{2\beta} a_{3\gamma} \cdots a_{n\omega} \)

\[ detA = a_{11}c_{11} + a_{12}c_{12} + \cdots + a_{1n}c_{1n} \]

- All terms including \( a_{11} \)
- All terms including \( a_{12} \)
- All terms including \( a_{1n} \)