Matrix-Vector Product

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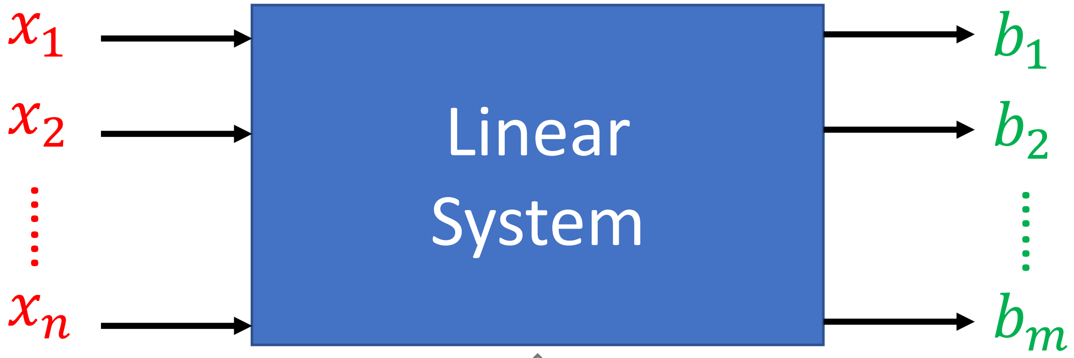


$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

Matrix-vector product: $A\mathbf{x} = \mathbf{b}$

Row Aspect

$$A = \begin{bmatrix} \boxed{a_{11} \quad a_{12} \quad \cdots \quad a_{1n}} \\ \boxed{a_{21} \quad a_{22} \quad \cdots \quad a_{2n}} \\ \vdots \quad\quad \vdots \quad\quad \ddots \quad\quad \vdots \\ \boxed{a_{m1} \quad a_{m2} \quad \cdots \quad a_{mn}} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

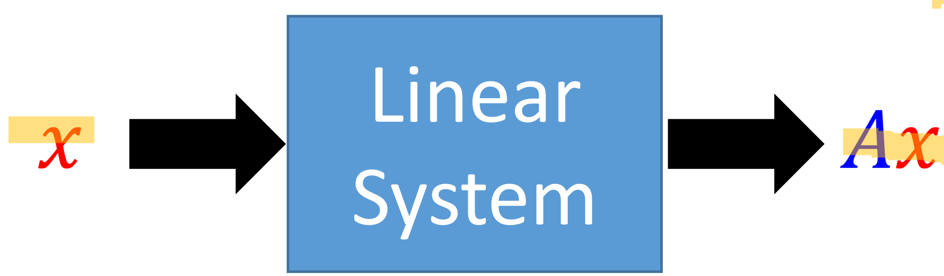
$$Ax = \begin{bmatrix} \boxed{a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n} \\ \boxed{a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n} \\ \vdots \\ \boxed{a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad \mathrm{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \mathrm{Ax} = \begin{bmatrix} \quad \end{bmatrix}$$

Matrix-Vector Product

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

$$Ax = b$$



Coefficients are A

Column Aspect

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

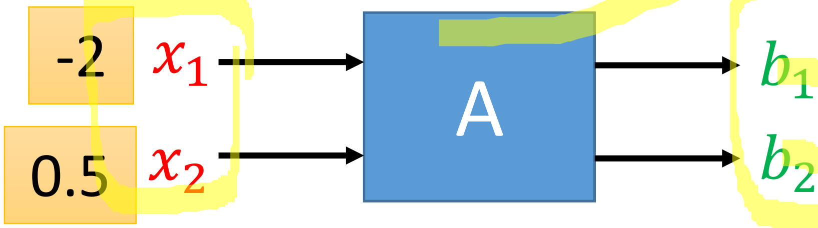
$$Ax =$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

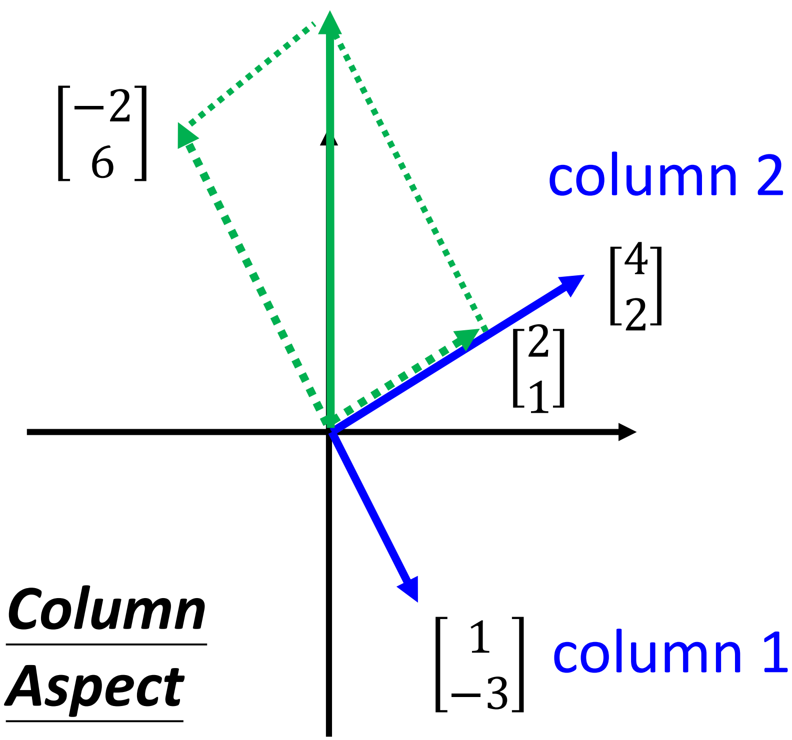
Example

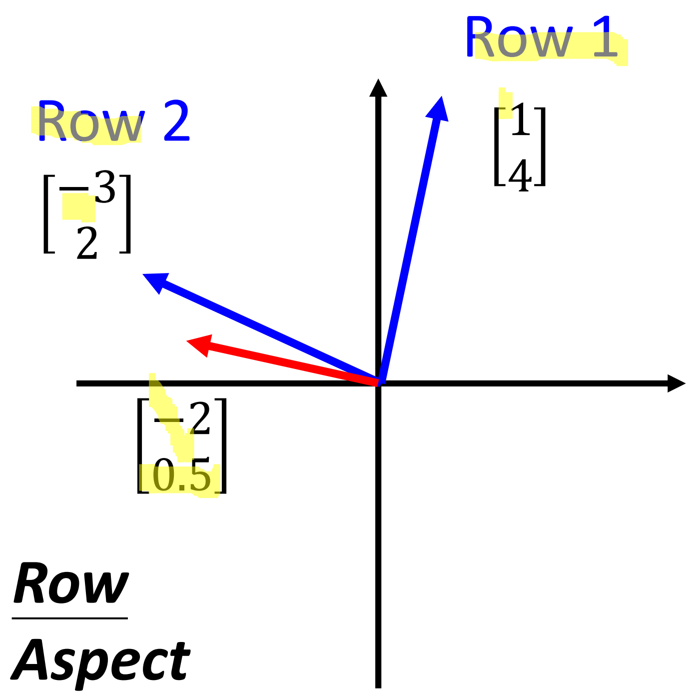
$$A = \begin{bmatrix} 1 & 4 \\ -3 & 2 \end{bmatrix}$$

$$x_1 + 4x_2 = b_1$$
$$-3x_1 + 2x_2 = b_2$$



$= Ax$





Matrix-vector Product

- The size of matrix and vector should be matched.

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 1 & -1 \\ -2 & 1 & 1 \end{bmatrix} \qquad x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A' = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 1 & 4 \end{bmatrix} \qquad A'' = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 0 & -1 \\ 1 & -3 \end{bmatrix}$$

Properties of Matrix-vector Product

- A and B are mxn matrices, **u** and **v** are vectors in $\mathcal{R}^n$, and c is a scalar.
- $A(\boldsymbol{u}+\boldsymbol{v}) = A\boldsymbol{u} + A\boldsymbol{v}$
- $A(c\boldsymbol{u}) = c(A\boldsymbol{u}) = (cA)\boldsymbol{u}$
- $(A+B)\boldsymbol{u} = A\boldsymbol{u} + B\boldsymbol{u}$
- $A\boldsymbol{0}$ is the mx1 zero vector
- $\boldsymbol{O}\boldsymbol{v}$ is also the mx1 zero vector
- $I_n\boldsymbol{v} = \boldsymbol{v}$

Properties of Matrix-vector Product

$$j \leftarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

- A and B are mxn matrices. If $A\boldsymbol{w} = B\boldsymbol{w}$ for all w in $\mathcal{R}^n$. Is it true that $A = B$?

$Ae_j = a_j$ for $j = 1,2,\cdots,n$, where e_j is the j-th standard vector in R^n

Column Aspect

$$e_1 = \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix} \qquad Ae_1 = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix} = 1 \cdot a_1 \cancel{+ 0 \cdot a_2 + \cdots + 0 \cdot a_n}$$

$$= a_1$$

$$Ae_1 = Be_1 \qquad Ae_2 = Be_2 \qquad \cdots\cdots \qquad Ae_n = Be_n$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad\qquad \Downarrow$$

$$a_1 = b_1 \qquad a_2 = b_2 \qquad \cdots\cdots \qquad a_n = b_n$$

$$\Rightarrow \quad A = B$$

Concluding Remarks

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

Row Aspect

$$= x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Column Aspect