Singular Value Decomposition
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Outline

• Diagonalization can only apply on some square matrices.
• Singular value decomposition (SVD) can apply on any matrix.

• Reference: Chapter 7.7
SVD

- Any $m \times n$ matrix $A$

\[
A = \begin{bmatrix}
U & \Sigma & V^T
\end{bmatrix}
\]

- $m \times m$ Orthonormal Set $U$
- $m \times n$ Diagonal $\Sigma$
- $n \times n$ Orthonormal Set $V^T$

- Orthonormal Set $U$ and $V^T$ are independent
- The diagonal entries of $\Sigma$ are non-negative
SVD
(We can exchange some rows and columns to achieve that)

• Any $m \times n$ matrix $A$

\[
A = \begin{bmatrix}
U & \Sigma & V^T
\end{bmatrix}
\]

$U$ is $m \times m$ Independent

$\Sigma$ is $m \times n$ Diagonal

$V^T$ is $n \times n$ Independent

If $A$ is a $m \times n$ matrix, and $B$ is a $n \times k$ matrix.

\[
Rank(AB) \leq \min(Rank(A), Rank(B))
\]

If $B$ is a matrix of rank $n$, then $Rank(AB) = Rank(A)$

If $A$ is a matrix of rank $n$, then $Rank(AB) = Rank(B)$
SVD

• Any $m \times n$ matrix $A$

$$A = U \Sigma V^T$$

Independent

Diagonal

Independent

$$A = U_1 \Sigma' V_1^T$$
SVD

\[
\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > 0
\]

\(\sigma_k\) is deleted

- Any \(m \times n\) matrix \(A\)

\[
A = \begin{bmatrix}
U & \Sigma \end{bmatrix}
\begin{bmatrix}
V^T
\end{bmatrix}
\]

\(\Sigma\) is Independent

\[
\begin{bmatrix}
\sigma_1 & 0 & \cdots & 0 \\
0 & \sigma_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \sigma_k
\end{bmatrix}
\]

What is the rank of \(A'\)?

\[
A' = \begin{bmatrix}
U_1 & \Sigma' \end{bmatrix}
\begin{bmatrix}
V_1^T
\end{bmatrix}
\]

\(A'\) is the rank \(k-1\) matrix minimizing \(\|A - A'\|\)
Low rank approximation using the singular value decomposition

https://www.youtube.com/watch?v=pAiVb7gWUrM
https://www.youtube.com/watch?v=fKVRSbFKnEw

It Had To Be U

The Singular Value Decomposition (SVD)
Thank You for Your Attention
https://www.youtube.com/watch?v=R9UoFyqJca8