Diagonalization

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Review

- If $Av = \lambda v$ (v is a vector, λ is a scalar)
  - v is an eigenvector of A excluding zero vector
  - λ is an eigenvalue of A that corresponds to v
- Eigenvectors corresponding to λ are nonzero solution of $(A - \lambda I_n)\mathbf{v} = \mathbf{0}$

Eigenvectors corresponding to λ

$$= \underbrace{Null(A - \lambda I_n)}_{\text{eigenspace}} - \{\mathbf{0}\}$$

Eigenspace of λ:

Eigenvectors corresponding to λ $+ \{\mathbf{0}\}$

- A scalar t is an eigenvalue of A

$$\Leftrightarrow \boxed{det(A - tI_n) = 0}$$

Review

- Characteristic polynomial of A is

$det(A - tI_n)$ Factorization

multiplicity

$$= (t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2} \dots (t-\lambda_k)^{m_k}\underline{(\dots\dots)}$$

Eigenvalue:	λ_1	λ_2	λ_k
Eigenspace: (dimension)	d_1 $\leq m_1$	d_2 $\leq m_2$	d_k $\leq m_k$

Outline

- An nxn matrix A is called diagonalizable if $A = PDP^{-1}$
  - D: nxn diagonal matrix
  - P: nxn invertible matrix
- Is a matrix A diagonalizable?
  - If yes, find D and P
- Reference: Textbook 5.3

Diagonalizable

- An nxn matrix A is called diagonalizable if $A = PDP^{-1}$
  - D: nxn diagonal matrix
  - P: nxn invertible matrix
- Not all matrices are diagonalizable

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $A^2 = 0$ (?)

If $A = PDP^{-1}$ for some invertible P and diagonal D

$\Rightarrow$ $A^2 = PD^2P^{-1} = 0$ $\Rightarrow$ $D^2 = 0$ $\Rightarrow$ $D = 0$

D is diagonal

$\Rightarrow$ A= 0 

Diagonalizable

$$P = [p_1 \quad \cdots \quad p_n]$$

$$D = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$

- If A is diagonalizable

$$A = PDP^{-1} \Rightarrow AP = PD$$

$$\Rightarrow AP = [\underline{Ap_1} \quad \cdots \quad \underline{Ap_n}]$$

$$\Rightarrow PD = P[d_1 e_1 \quad \cdots \quad d_n e_n]$$

$$= [Pd_1 e_1 \quad \cdots \quad Pd_n e_n]$$

$$= [d_1 P e_1 \quad \cdots \quad d_n P e_n]$$

$$= [\underline{d_1 p_1} \quad \cdots \quad \underline{d_n p_n}] \Rightarrow Ap_i = d_i p_i$$

p_i is an eigenvector of A corresponding to eigenvalue d_i

Diagonalizable

$$P = [p_1 \quad \cdots \quad p_n]$$

$$D = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$

- If A is diagonalizable

$$A = PDP^{-1}$$

p_i is an eigenvector of A corresponding to eigenvalue d_i

=

There are n eigenvectors that form an invertible matrix

=

There are n independent eigenvectors

=

The eigenvectors of A can form a basis for R^n.

Diagonalizable

$$P = [p_1 \quad \cdots \quad p_n]$$

$$D = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$

- If A is diagonalizable

$$A = PDP^{-1}$$

p_i is an eigenvector of A corresponding to eigenvalue d_i

How to diagonalize a matrix A?

Step 1: Find *n* independent eigenvectors corresponding if possible, and form an invertible *P*

Step 2: The eigenvalues corresponding to the eigenvectors in P form the diagonal matrix *D.*

Diagonalizable

A set of eigenvectors that correspond to distinct eigenvalues is linear independent.

$det(A - tI_n)$ Factorization

$= (t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2} \ldots (t-\lambda_k)^{m_k}(\ldots\ldots)$

Eigenvalue: λ_1 λ_2 …… λ_k

Eigenspace: $d_1 \leq m_1$ $d_2 \leq m_2$ …… $d_k \leq m_k$

(dimension)

Independent

Diagonalizable

A set of eigenvectors that correspond to distinct eigenvalues is linear independent.

Eigenvalue: λ_1 λ_2 …… λ_m

Eigenvector: v_1 v_2 …… v_m

Assume dependent $\Rightarrow$ a contradiction

$$\mathbf{v}_k = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{k-1}\mathbf{v}_{k-1}$$

$$A\mathbf{v}_k = c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + \cdots + c_{k-1}A\mathbf{v}_{k-1}$$

$$\lambda_k\mathbf{v}_k = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + \cdots + c_{k-1}\lambda_{k-1}\mathbf{v}_{k-1}$$

$$-\quad \lambda_k\mathbf{v}_k = c_1\lambda_k\mathbf{v}_1 + c_2\lambda_k\mathbf{v}_2 + \cdots + c_{k-1}\lambda_k\mathbf{v}_{k-1}$$

(λ_k)

$$\mathbf{0} = c_1(\lambda_1 - \lambda_k)\mathbf{v}_1 + c_2(\lambda_2 - \lambda_k)\mathbf{v}_2 + \cdots + c_{k-1}(\lambda_{k-1} - \lambda_k)\mathbf{v}_{k-1}$$

Not $c_1 = c_2 = \cdots = c_{k-1} = 0$ $\Rightarrow$ *Same eigenvalue* $\Rightarrow$ a contradiction

$$P = [p_1 \quad \cdots \quad p_n]$$

Diagonalizable

$$D = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$

- If A is diagonalizable

$$A = PDP^{-1}$$

p_i is an eigenvector of A corresponding to eigenvalue d_i

$$det(A - tI_n)$$
$$= (t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2} \ldots (t-\lambda_k)^{m_k}(\ldots\ldots)$$

Eigenvalue: λ_1 λ_2 ...... λ_k

Eigenspace: ......

$\{$ Basis for λ_1 Basis for λ_2 Basis for λ_3 $\}$

Independent Eigenvectors

You can't find more!

Diagonalizable - Example

- Diagonalize a given matrix $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

characteristic polynomial is $-(t+1)^2(t-3)$ ⇒ eigenvalues: 3, –1

eigenvalue 3

$$\mathbb{B}_1 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

eigenvalue –1

$$\mathbb{B}_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

$A = PDP^{-1}$, where

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

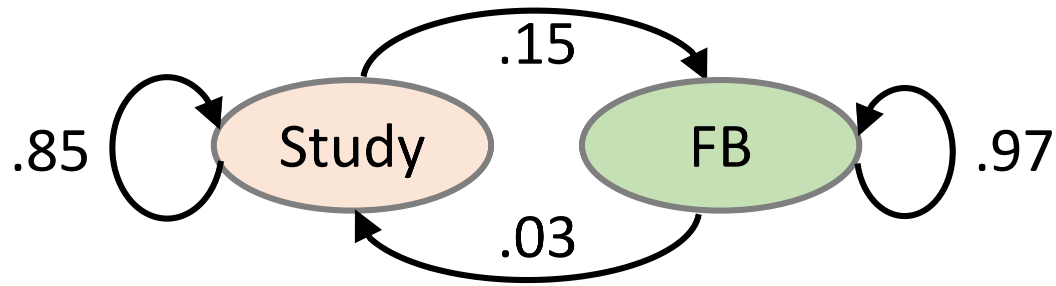
$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

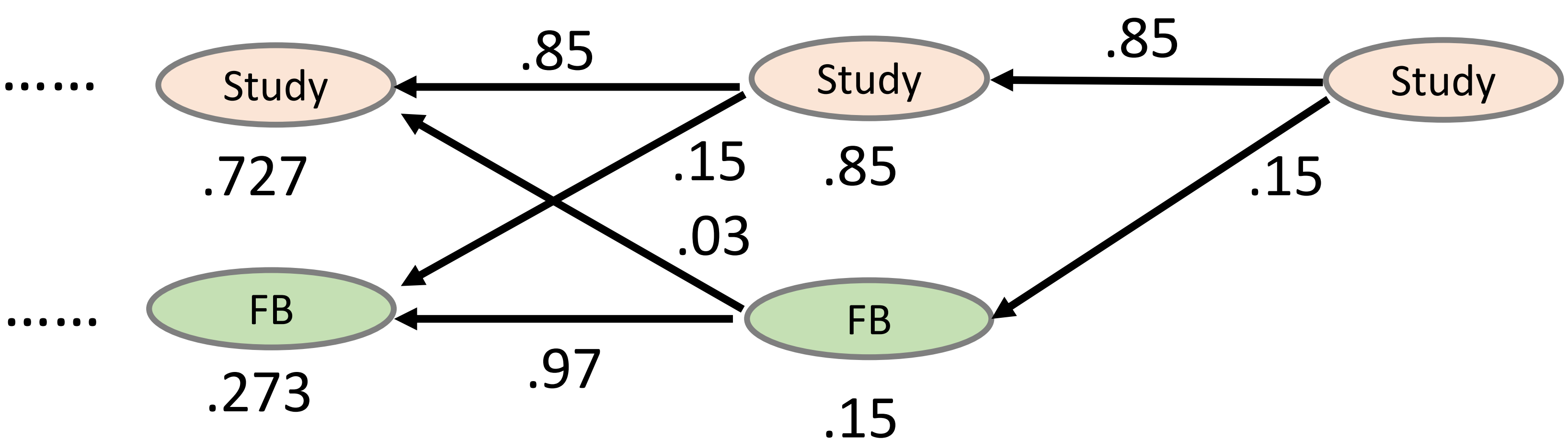
Application of Diagonalization

- If A is diagonalizable,

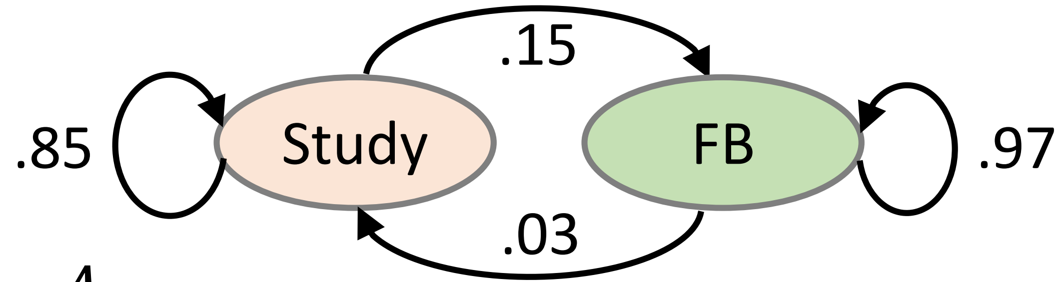
$$A = PDP^{-1} \Longrightarrow A^m = PD^mP^{-1}$$

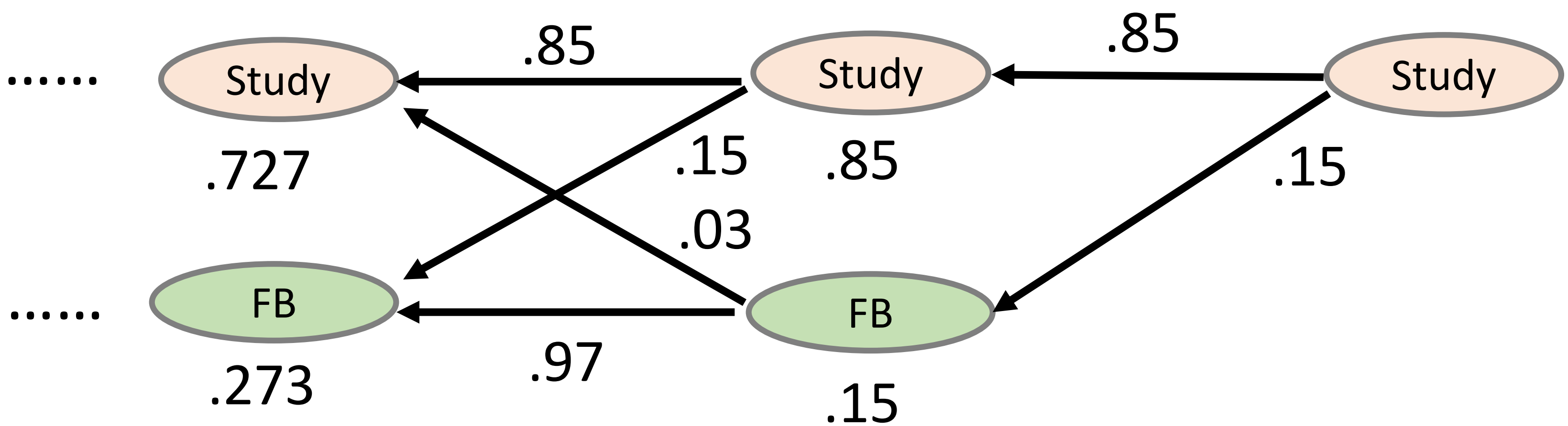
- Example:





$$
\begin{array}{ccc}
 & & \text{From} \\
 & & \begin{array}{cc}\text{Study} & \text{FB}\end{array} \\
\text{To} & \begin{array}{c}\text{Study}\\ \text{FB}\end{array} & \begin{bmatrix} .85 & .03 \\ .15 & .97 \end{bmatrix} = A
\end{array}
$$





$$
\begin{bmatrix} .727 \\ .273 \end{bmatrix} \quad \begin{bmatrix} .85 & .03 \\ .15 & .97 \end{bmatrix} \quad \begin{bmatrix} .85 \\ .15 \end{bmatrix} \quad \begin{bmatrix} .85 & .03 \\ .15 & .97 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}
$$

$$
A = PDP^{-1} \Longrightarrow A^m = PD^mP^{-1}
$$

Diagonalizable

- Diagonalize a given matrix $A = \begin{bmatrix} .85 & .03 \\ .15 & .97 \end{bmatrix}$

$$\det\ (A - tI_2)$$

$$A - .82I_2 \xrightarrow[\text{RREF}]{} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$A - I_2 \xrightarrow[\text{RREF}]{} \begin{bmatrix} 1 & -.2 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{p}_2 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & 1 \\ 1 & 5 \end{bmatrix}$$

(invertible)

$$D = \begin{bmatrix} .82 & 0 \\ 0 & 1 \end{bmatrix}$$

Application of Diagonalization

$$A = PDP^{-1} \text{ where } P = \begin{bmatrix} -1 & 1 \\ 1 & 5 \end{bmatrix}, D = \begin{bmatrix} .82 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} A^m &= PD^mP^{-1} \\ &= \begin{bmatrix} -1 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} (.82)^m & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 5 \end{bmatrix}^{-1} \\ &= \frac{1}{6} \begin{bmatrix} 1+5(.82)^m & 1-(.82)^m \\ 5-5(.82)^m & 5+(.82)^m \end{bmatrix} \end{aligned}$$

When $m \to \infty$,

$$A^m = \begin{bmatrix} 1/6 & 1/6 \\ 5/6 & 5/6 \end{bmatrix}$$

The beginning condition does not influence.

Test for a Diagonalizable Matrix

- An *n* x *n* matrix *A* is diagonalizable if and only if both the following conditions are met.

The characteristic polynomial of *A* factors into a product of linear factors.

$$det(A - tI_n)$$

Factorization

$$= (t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2} \dots (t-\lambda_k)^{m_k}\cancel{(\dots\dots)}$$

For each eigenvalue λ of *A*, the multiplicity of λ equals the dimension of the corresponding eigenspace.

Independent Eigenvectors

An n x n matrix A is diagonalizable

||

The eigenvectors of A can form a basis for R^n.

||

$$det(A - tI_n)$$

$$= (t-\lambda_1)^{\cancel{m_1}}(t-\lambda_2)^{\cancel{m_2}} \ldots (t-\lambda_k)^{\cancel{m_k}}\cancel{(\ldots\ldots)}$$

Eigenvalue:	λ_1	λ_2	......	λ_k
Eigenspace: (dimension)	$d_1 = m_1$	$d_2 = m_2$	......	$d_k = m_k$

Diagonalization of Linear Operator

- Example 1: $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 8x_1 + 9x_2 \\ -6x_1 - 7x_2 \\ 3x_1 + 3x_2 - x_3 \end{bmatrix}$

$\boxed{\text{det}}$

The standard matrix is $A = \begin{bmatrix} 8\ \boxed{-t} & 9 & 0 \\ -6 & -7\ \boxed{-t} & 0 \\ 3 & 3 & -1\ \boxed{-t} \end{bmatrix}$

$\Rightarrow$ the characteristic polynomial is $-(t + 1)^2(t - 2)$

Eigenvalue -1: Eigenvalue 2:

$\mathcal{B}_1 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ $\mathcal{B}_2 = \left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \right\}$ $\Rightarrow \mathcal{B}_1 \cup \mathcal{B}_2$ is a basis of $\mathcal{R}^3$

Diagonalization of Linear Operator

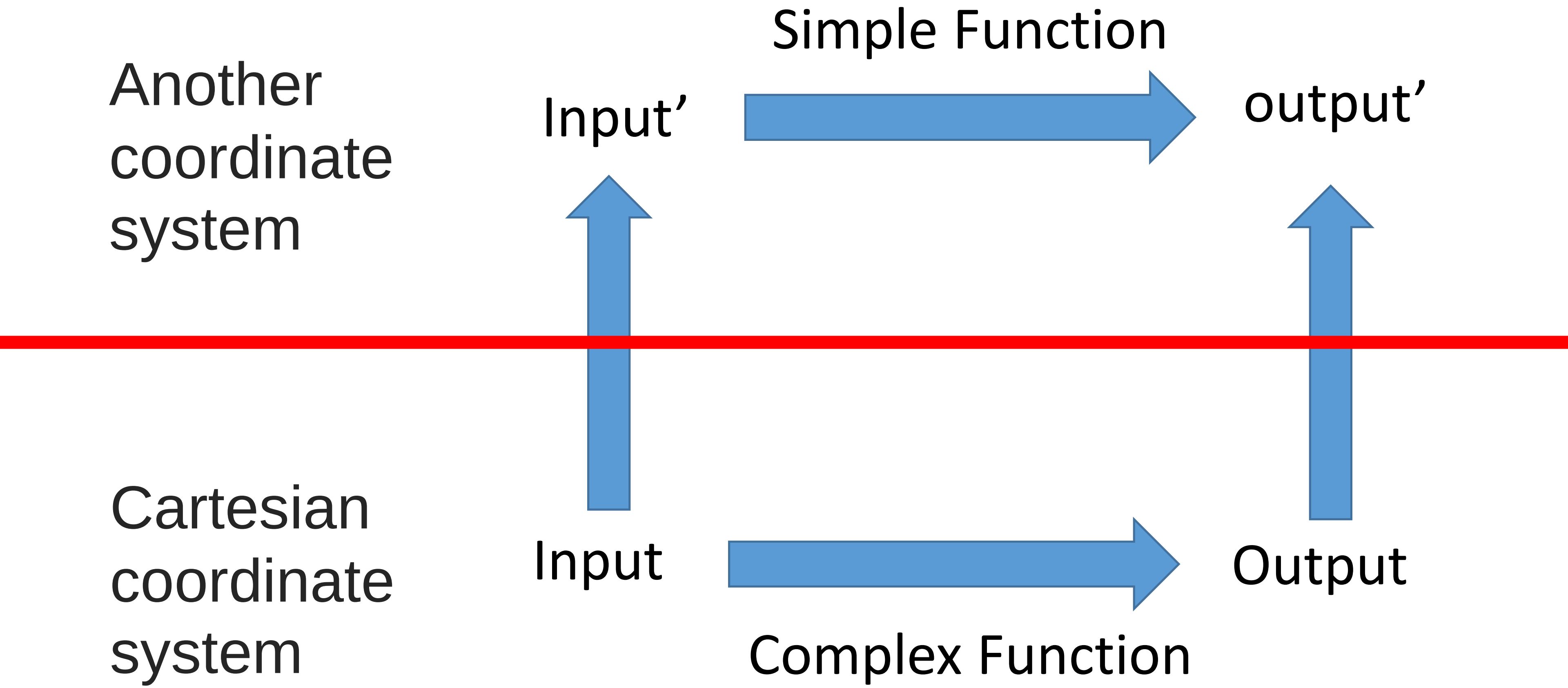
- Example 2: $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_1 + x_2 + 2x_3 \\ x_1 - x_2 \\ 0 \end{bmatrix}$

det -t -t -t

The standard matrix is $A = \begin{bmatrix} -1 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Review


Another coordinate system
Cartesian coordinate system
Simple Function
Input’
output’
Input
Output
Complex Function

Flowchart

$\mathcal{B}$ coordinate system

Cartesian coordinate system

$$
\begin{tikzcd}
{[v]_{\mathcal{B}}} \arrow[rr, "{[T]_{\mathcal{B}}}"] & & {[T(v)]_{\mathcal{B}}} \arrow[d, "B"] \\
v \arrow[u, "B^{-1}"] \arrow[rr, "{[T]}"'] & & T(v)
\end{tikzcd}
$$

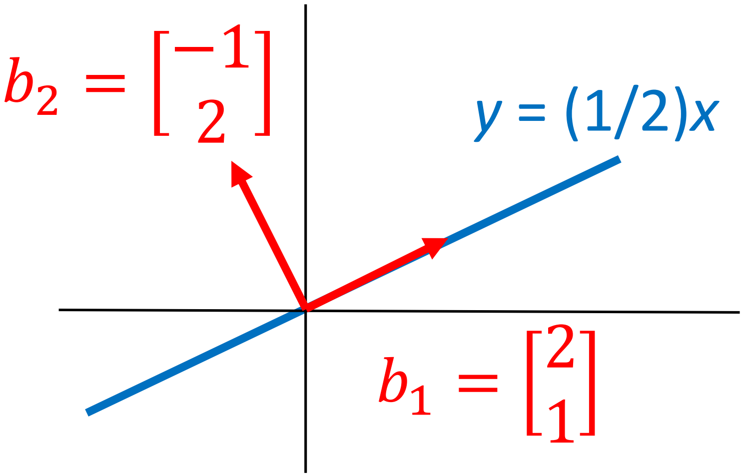
$$[T] = B[T]_{\mathcal{B}}B^{-1}$$

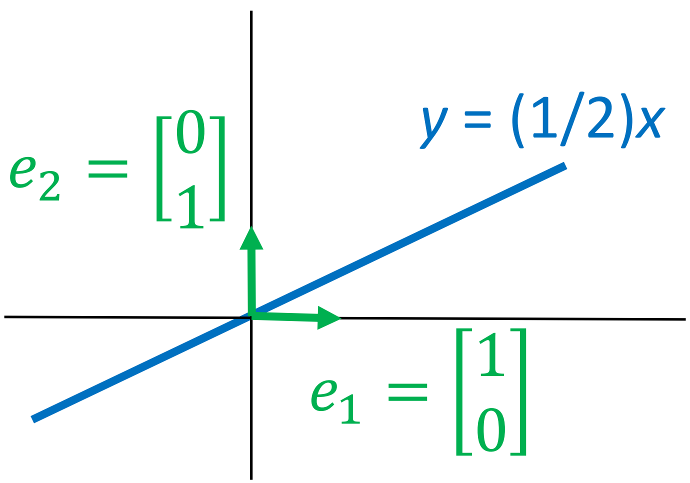
similar

$$[T]_{\mathcal{B}} = B^{-1}[T]B$$

similar

- Example: reflection operator T about the line $y = (1/2)x$





$$B^{-1} = \begin{bmatrix} 0.4 & 0.2 \\ -0.2 & 0.4 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\begin{tikzcd}
{[v]_{\mathcal{B}}} \arrow[r, "{[T]_{\mathcal{B}}}", "{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}"'] & {[T(v)]_{\mathcal{B}}} \arrow[d, "B"] \\
v \arrow[u, "B^{-1}"'] \arrow[r, "{[T] =?}"] & T(v)
\end{tikzcd}$$

$$[T] = B[T]_{\mathcal{B}}B^{-1}$$

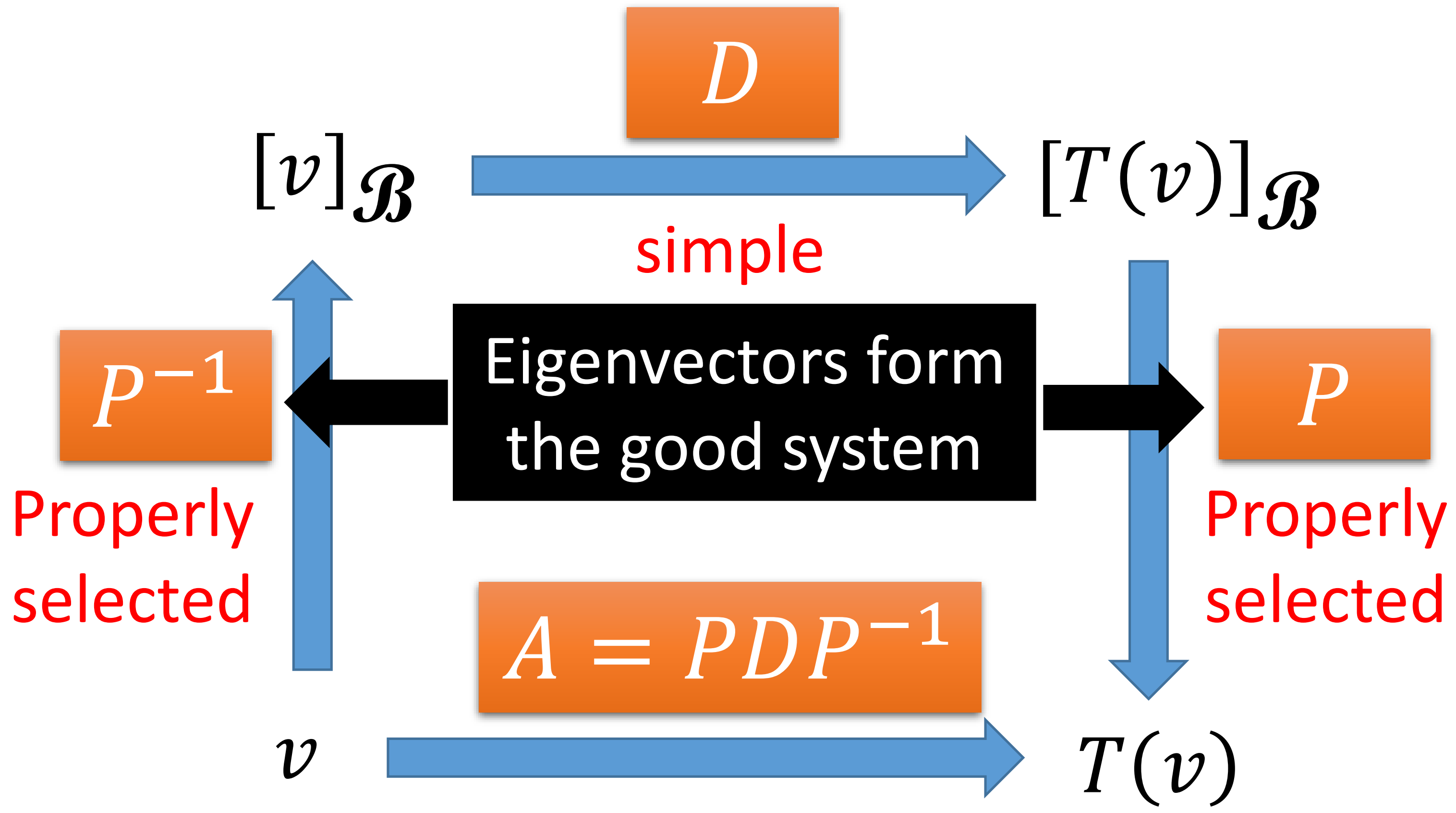
Diagonalization of Linear Operator

- Reference: Chapter 5.4

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
{[v]_{\mathcal{B}}} \arrow[r, "{[T]_{\mathcal{B}}}", "\text{simple}"'] & {[T(v)]_{\mathcal{B}}} \arrow[d, "{\begin{matrix} B \\ \text{Properly selected} \end{matrix}}"] \\
v \arrow[u, "{\begin{matrix} B^{-1} \\ \text{Properly selected} \end{matrix}}"] \arrow[r, "T"] & T(v)
\end{tikzcd}
$$

Diagonalization of Linear Operator

- If a linear operator T is diagonalizable



Diagonalization of Linear Operator

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 8x_1 + 9x_2 \\ -6x_1 - 7x_2 \\ 3x_1 + 3x_2 - x_3 \end{bmatrix}$$

-1: $\mathcal{B}_1 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

2: $\mathcal{B}_2 = \left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \right\}$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
{[v]_{\mathcal{B}}} \arrow[r, "{[T]_{\mathcal{B}}}", "{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}"'] & {[T(v)]_{\mathcal{B}}} \arrow[d, "{\begin{bmatrix} -1 & 0 & 3 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}}"] \\
v \arrow[u, "{\begin{bmatrix} -1 & 0 & 3 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}^{-1}}"] \arrow[r, "{\begin{bmatrix} 8 & 9 & 0 \\ -6 & -7 & 0 \\ 3 & 3 & -1 \end{bmatrix}}"] & T(v)
\end{tikzcd}$$