

Orthogonality

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Outline

- Reference: Chapter 7.1

Norm & Distance

- **Norm:** Norm of vector v is the length of v

- Denoted $\|v\|$

$$\|v\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

- **Distance:** The distance between two vectors u and v is defined by $\|v - u\|$

$$v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad u = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix} \quad v - u = \begin{bmatrix} -1 \\ 5 \\ 3 \end{bmatrix}$$
$$\|v\| = \sqrt{1^2 + 2^2 + 3^2}$$
$$= \sqrt{14}$$
$$\|v - u\| = \sqrt{(-1)^2 + 5^2 + 3^2}$$
$$= \sqrt{35}$$

$$\|v\|_p = \left(\sum_{i=1}^n |v_i|^p \right)^{1/p}$$

$$\|v\|_1 = |v_1| + |v_2| + \cdots + |v_n|$$

$$\|v\|_2 = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

$$\|v\|_3 = \sqrt[3]{|v_1|^3 + |v_2|^3 + \cdots + |v_n|^3}$$

$$\|v\|_4 = \sqrt[4]{v_1^4 + v_2^4 + \cdots + v_n^4}$$

Orthogonal



<https://www.youtube.com/watch?v=43BfcSkctYA>

<https://www.youtube.com/watch?v=EktZVposDMU>

Dot Product & Orthogonal

- **Dot product:** dot product of u and v is

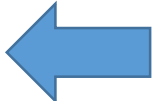
$$\begin{aligned} u \cdot v &= u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \\ &= \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u^T v \end{aligned}$$

- **Orthogonal:** u and v are orthogonal if $u \cdot v = 0$

Orthogonal is actually “perpendicular”

Zero vector is orthogonal to every vector

More about Dot Product

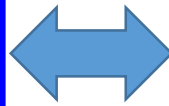
- Let u and v be vectors, A be a matrix, and c be a scalar
- $u \cdot u = \|u\|^2$  **Connect norm and dot product**
- $u \cdot u = 0$ if and only if $u = 0$
- $u \cdot v = v \cdot u$
- $u \cdot (v + w) = u \cdot v + u \cdot w$
- $(v + w) \cdot u = v \cdot u + w \cdot u$
- $(cu) \cdot v = c(u \cdot v) = u \cdot (cv)$
- $\|cu\| = |c|\|u\|$
- $Au \cdot v$

Example $\|2\mathbf{u} + 3\mathbf{v}\|^2 = \dots$

$$= 4\|\mathbf{u}\|^2 + 12(\mathbf{v} \cdot \mathbf{u}) + 9\|\mathbf{v}\|^2.$$

Pythagorean Theorem

\mathbf{u} and \mathbf{v} are orthogonal



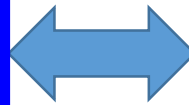
$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Proof: $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \underline{2\mathbf{u} \cdot \mathbf{v}} + \|\mathbf{v}\|^2$

=0 if and only if \mathbf{u}
and \mathbf{v} are orthogonal

Pythagorean Theorem

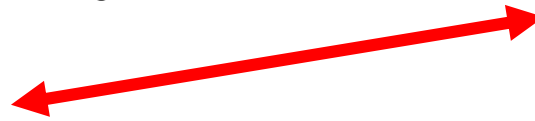
The diagonals of a parallelogram are orthogonal.



The parallelogram is a rhombus (菱形).

Proof: $(u + v) \cdot (u - v) = 0$
 $= \|u\|^2 - \|v\|^2$

$\|u\| = \|v\|$



Triangle Inequality

- For any vectors u and v ,

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

Proof: $\|u + v\|^2 = \|u\|^2 + 2u \cdot v + \|v\|^2$
 $\leq \|u\|^2 + 2|u \cdot v| + \|v\|^2$

Cauchy-Schwarz Inequality $\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2$
 $\leq (\|u\| + \|v\|)^2$

Proof:

$$\begin{aligned}\|u + v\|^2 &= \|u\|^2 + 2u \cdot v + \|v\|^2 \\ &\leq \|u\|^2 + 2|u \cdot v| + \|v\|^2\end{aligned}$$

$$\begin{aligned}\text{Cauchy-Schwarz Inequality} &\leq \|u\|^2 + 2\|u\| \cdot \|v\| + \|v\|^2 \\ &\leq (\|u\| + \|v\|)^2\end{aligned}$$