# Having Solution or Not

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# Reference

- Textbook: Chapter 1.6

# Learning Target

***Review***


$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

*System of Linear Equations*

Matrix-vector product: $A\mathbf{x} = \mathbf{b}$

- Given A and **b**, sometimes **x** exists, and sometimes doesn't.
- A system of linear equations has solution or not.
- New terms: "*linear combination*" and "*span*"

# Solution

- A system of linear equations is called consistent if it has one or more solutions.
- A system of linear equations is called inconsistent if its solution set is empty (no solution).

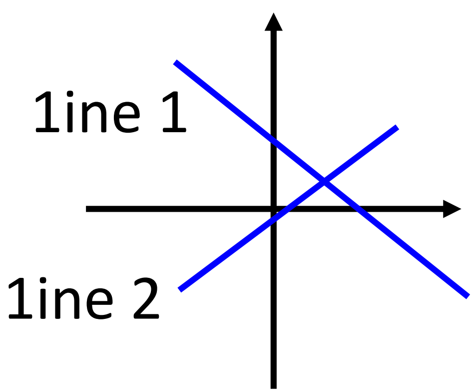
| | $\begin{aligned} 3x_1 + x_2 &= 10 \\ x_1 - 3x_2 &= 0 \end{aligned}$ | $\begin{aligned} 3x_1 + x_2 &= 10 \\ 6x_1 + 2x_2 &= 20 \end{aligned}$ | $\begin{aligned} 3x_1 + x_2 &= 10 \\ 6x_1 + 2x_2 &= 0 \end{aligned}$ |
|---|---|---|---|
| Solution set | $\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$ | $\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 3 \end{bmatrix} : \forall t \in \mathcal{R} \right\}$ | $\{\}$, or $\phi$ |
| Consistent or Inconsistent? | Consistent | Consistent | Inconsistent |

# Solution (High School)

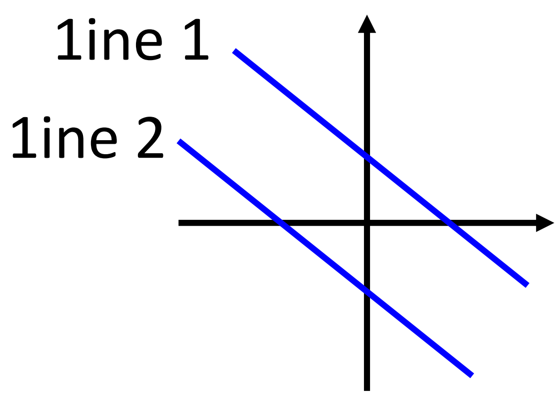
More Variables?

- Considering any system of linear equations with 2 variables and 2 equations

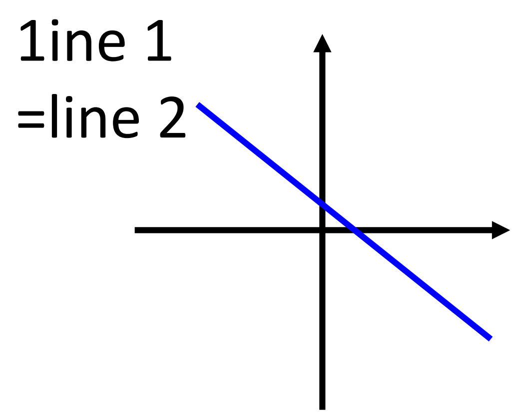
$$\color{blue}{a_{11}}\color{red}{x_1} + \color{blue}{a_{12}}\color{red}{x_2} = \color{green}{b_1} \qquad \text{...... line 1}$$
$$\color{blue}{a_{21}}\color{red}{x_1} + \color{blue}{a_{22}}\color{red}{x_2} = \color{green}{b_2} \qquad \text{...... line 2}$$



unique solution



no solution



infinitely many solution

# Linear Combination

# Linear Combination

- Given a vector set $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_k\}$
- The linear combination of the vectors in the set:
  - $\boldsymbol{v} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k$
  - $c_1, c_2, \cdots, c_k$ are scalars (Coefficients of linear combination)

vector set: $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}$

coefficients: $\{-3,4,1\}$

What is the result of linear combination?

# Column Aspect

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
 & & & & \vdots & & & & \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \\
\boldsymbol{a}_1 & & \boldsymbol{a}_2 & & & & \boldsymbol{a}_n & &
\end{array}
$$

$$A = [\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \cdots \quad \boldsymbol{a}_n]$$

Vector set

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

coefficients

$$A\mathbf{x} = x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n$$

Linear Combination

# System of Linear Equations v.s. Linear Combination

$$A\mathbf{x} = \mathbf{b}$$

(A system of linear equations)

Non empty solution set?

Has solution or not?

Consistent?

The Same question

***Column Aspect***

$$A\mathbf{x}$$

$$= x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = b$$

the linear combination of columns of $A$

Is $b$ the linear combination of columns of $A$?

# Example 1

$$3x_1 + 6x_2 = 3$$
$$2x_1 + 4x_2 = 4$$

Has solution or not?

$$A\mathbf{x} = \mathbf{b}$$

$$A = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Is $b$ the linear combination of columns of $A$?

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \end{bmatrix} \right\}$$

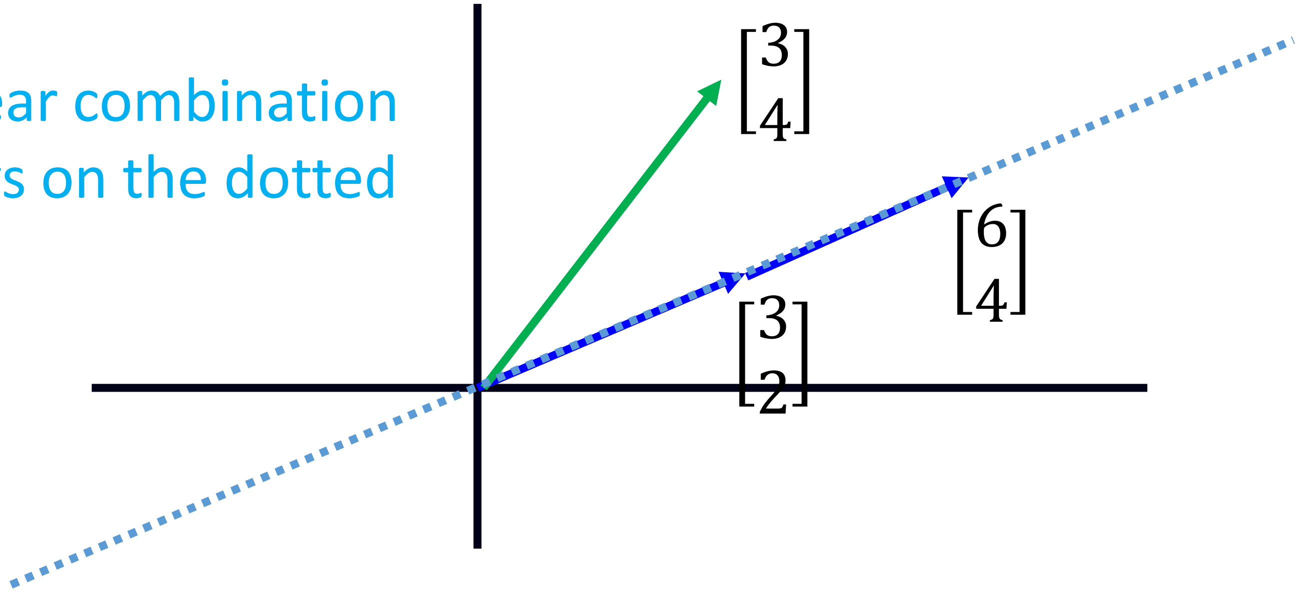
# Example 1

$$3x_1 + 6x_2 = 3$$
$$2x_1 + 4x_2 = 4$$

Has solution or not?

- Vector set: $\left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \end{bmatrix} \right\}$
- Is $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ a linear combination of $\left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \end{bmatrix} \right\}$? No

The linear combination is always on the dotted line.



# Example 2

$$2x_1 + 3x_2 = 4$$
$$3x_1 + 1x_2 = -1$$

Has solution or not?

$$A\mathbf{x} = \mathbf{b}$$

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

Is $b$ the linear combination of columns of $A$?

$$\begin{bmatrix} 4 \\ -1 \end{bmatrix} \qquad \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$$

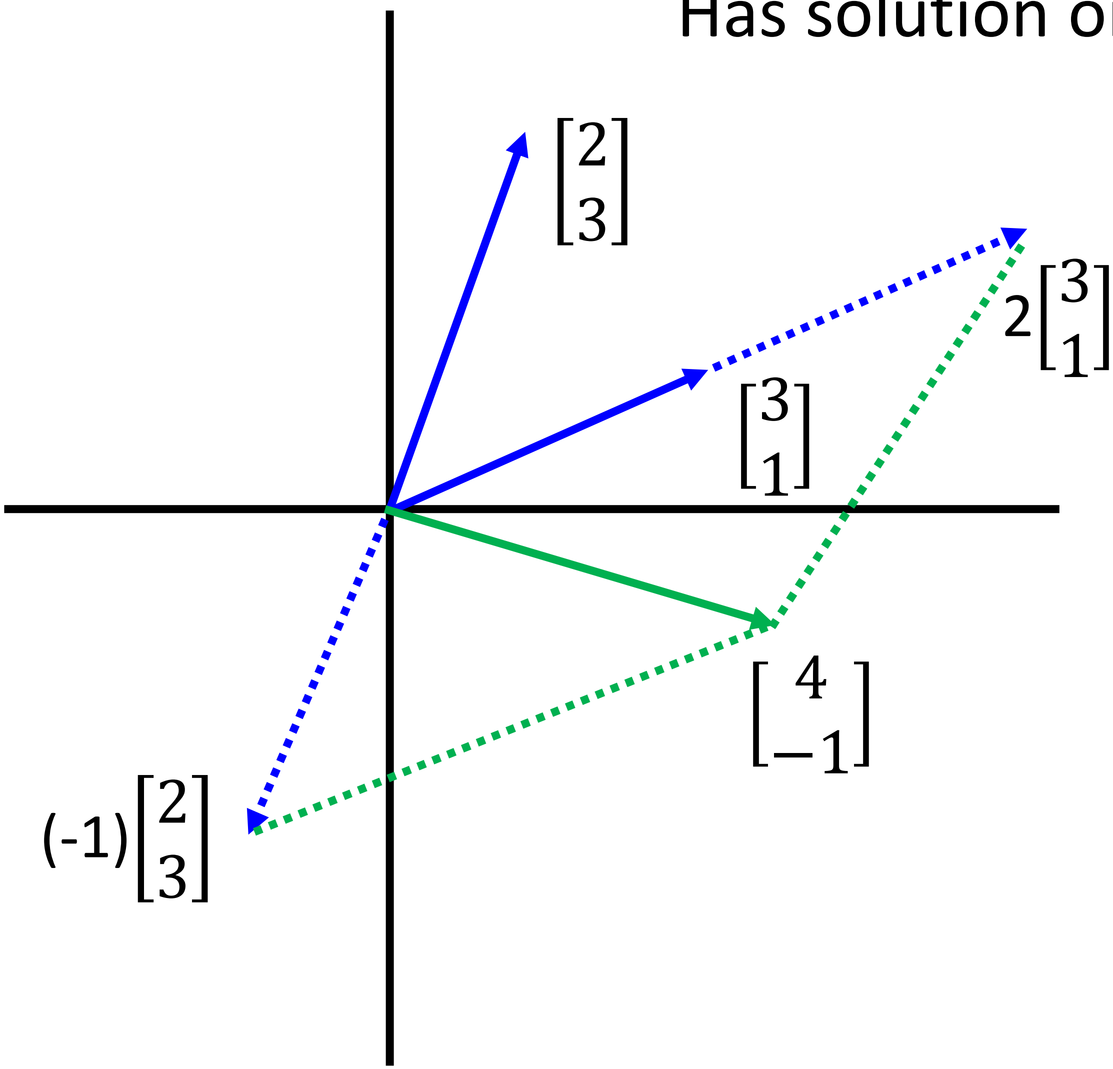
# Example 2

$$\left\{\begin{bmatrix}2\\3\end{bmatrix},\begin{bmatrix}3\\1\end{bmatrix}\right\}$$

$$\begin{bmatrix}4\\-1\end{bmatrix}$$

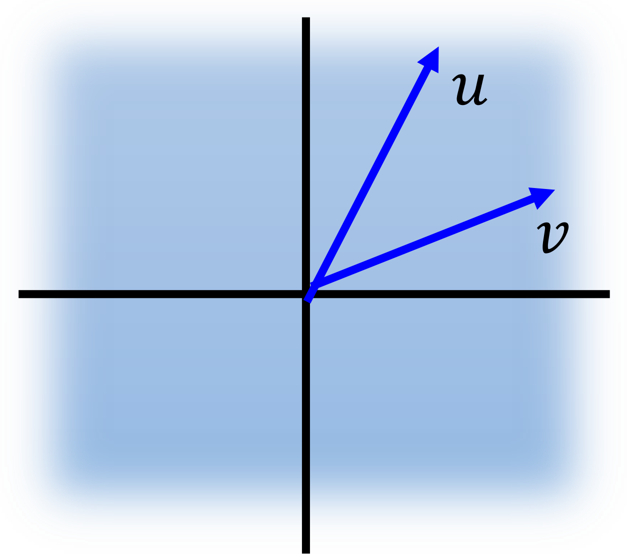
$$\boxed{\begin{aligned}2x_1 + 3x_2 &= 4\\ 3x_1 + 1x_2 &= -1\end{aligned}}$$

Has solution or not?



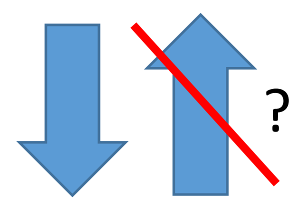
# Example 2

- If **u** and **v** are any nonparallel vectors in $\mathscr{R}^2$, then every vector in $\mathscr{R}^2$ is a linear combination of **u** and **v**
    - Nonparallel: **u** and **v** are nonzero vectors, and **u** $\neq c$**v**.




$$\begin{aligned} u_1x_1 \quad &+v_1x_2 \quad &= b_1 \\ u_2x_1 \quad &+v_2x_2 \quad &= b_2 \end{aligned}$$

**u** and **v** are not parallel




Has solution

- If **u, v** and **w** are any nonparallel vectors in $\mathscr{R}^3$, then every vector in $\mathscr{R}^3$ is a linear combination of **u, v** and **w**?

# Example 3

$$2x_1 + 6x_2 = -4$$
$$1x_1 + 3x_2 = -2$$

Has solution or not?

$$A\mathbf{x} = \mathbf{b}$$

$$A = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad b = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$$

Is $b$ the linear combination of columns of $A$?

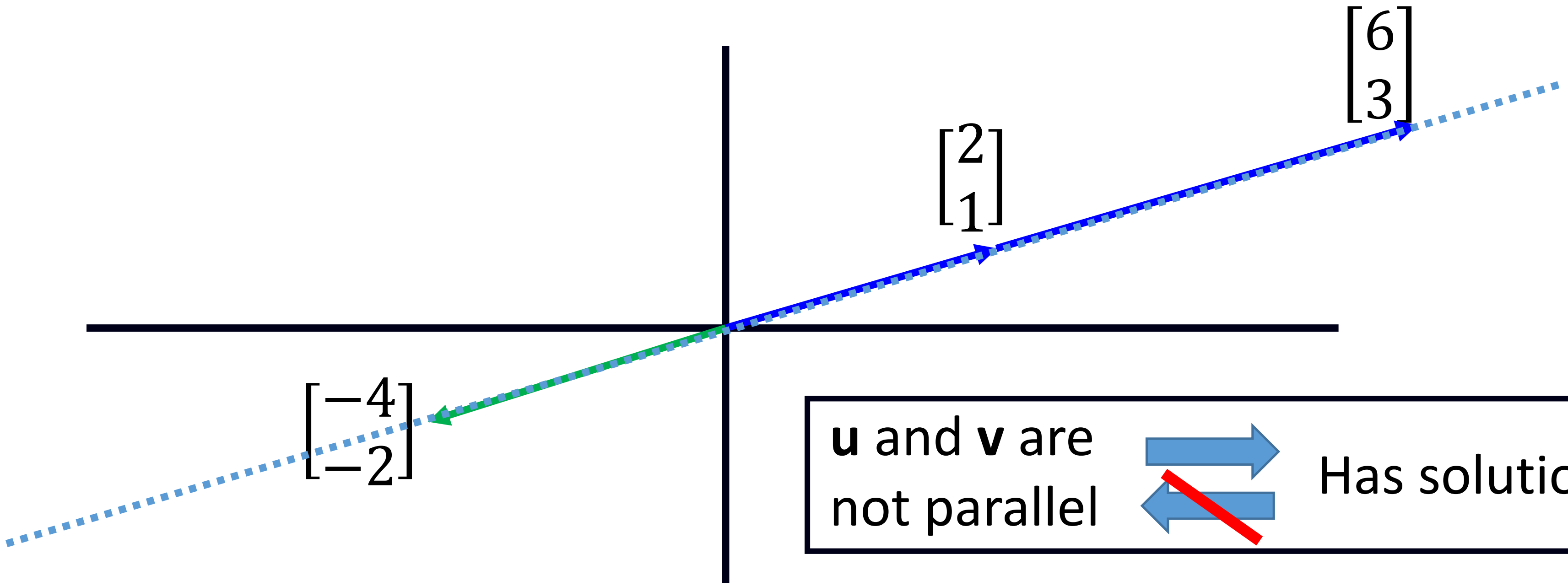
$$\begin{bmatrix} -4 \\ -2 \end{bmatrix} \qquad \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \end{bmatrix} \right\}$$

# Example 3

$$2x_1 + 6x_2 = -4$$
$$1x_1 + 3x_2 = -2$$

Has solution or not?

- Vector set: $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \end{bmatrix} \right\}$
- Is $\begin{bmatrix} -4 \\ -2 \end{bmatrix}$ a linear combination of $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \end{bmatrix} \right\}$? Yes




**u** and **v** are not parallel $\rightleftarrows$ Has solution

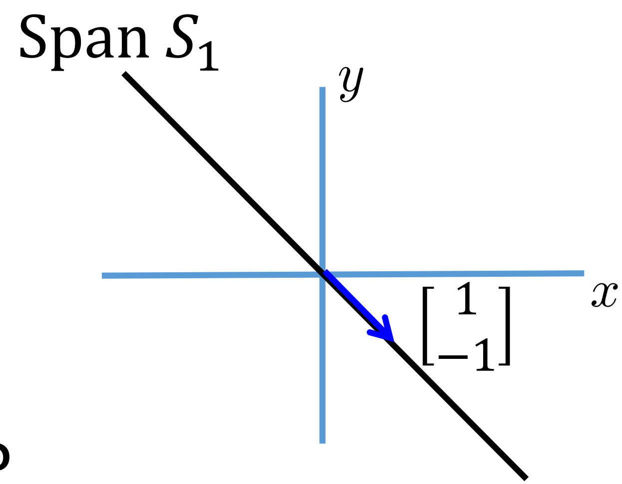
# Span

# Span

- A vector set $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_k\}$
- $Span\ S$ is the vector set of all linear combinations of $\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_k$
  - $Span\ S = \{c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k |\ for\ all\ c_1, c_2, \cdots, c_k\}$
- Vector set $V = Span\ S$
  - "$S$ is a generating set for $V$" or "$S$ generates V"
  - One way to describe a vector set (with infinite elements)

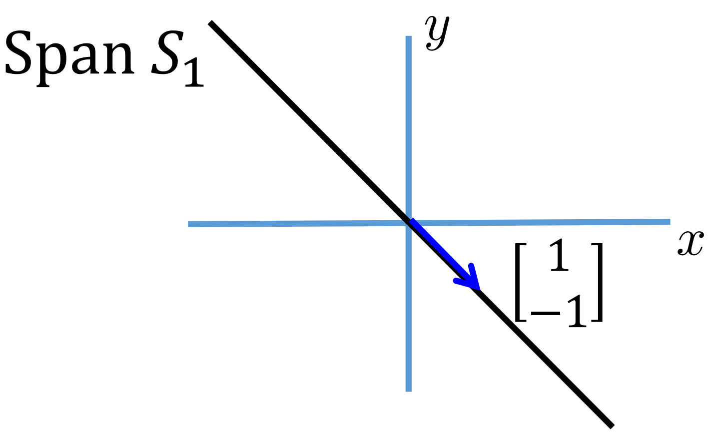
# Span

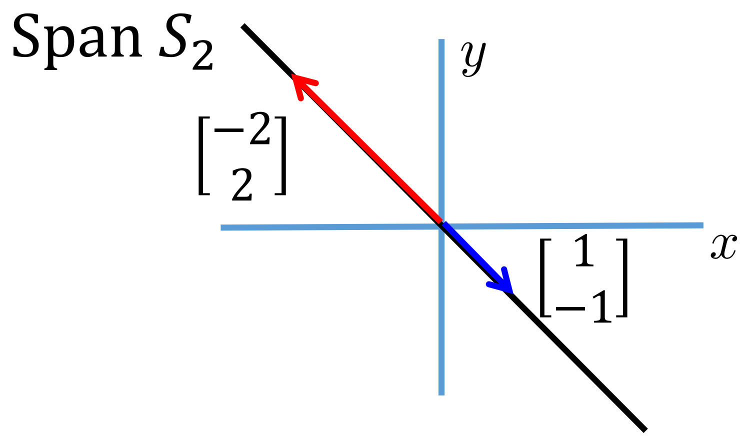
- Let $S_0 = \{\begin{bmatrix}0\\0\end{bmatrix}\}$, what is $\text{Span } S_0$?
  - Ans: $\{\begin{bmatrix}0\\0\end{bmatrix}\}$ (only one member)
- Let $S_1 = \{\begin{bmatrix}1\\-1\end{bmatrix}\}$, what is $\text{Span } S_1$?
  - If S contains a non zero vector, then Span S has infinitely many vectors



# Span

- Let $S_1 = \{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \}$, what is Span $S_1$?
- Let $S_2 = \{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix} \}$, what is Span $S_2$?



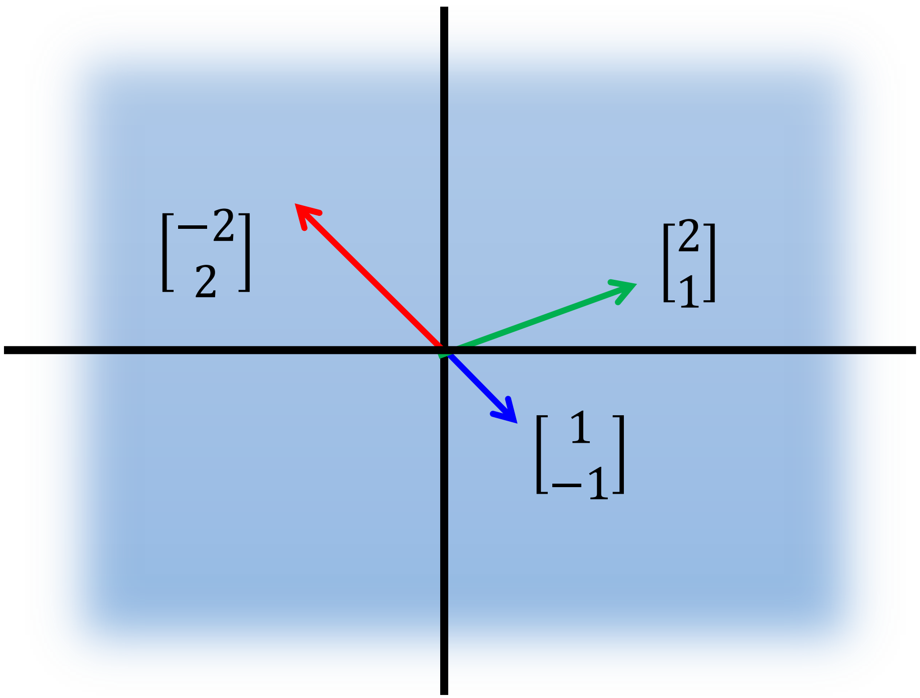

$$\text{Span } S_1 = \text{Span } S_2$$

(Different number of vectors can generate the same space.)

# Span

- Let $S_3 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$, what is Span $S_3$?

nonparallel vectors

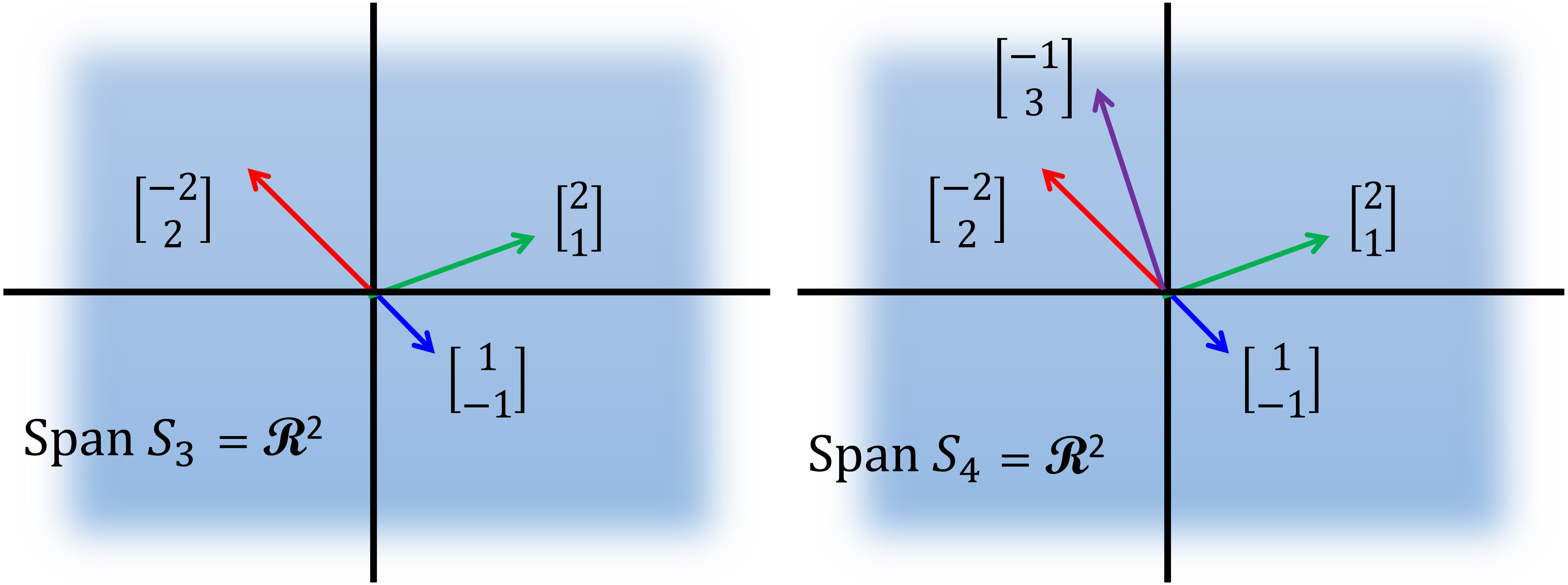



Every vector in $\mathcal{R}^2$ is their linear combination

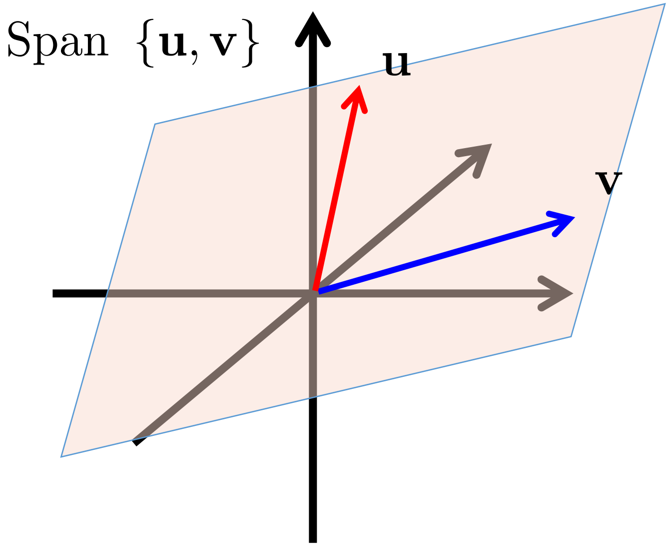
$$\text{Span } S_3 = \mathcal{R}^2$$

# Span

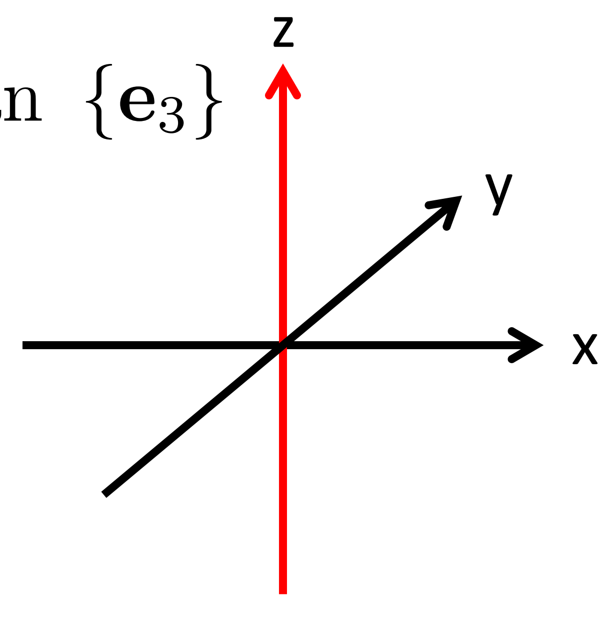
- Let $S_3 = \{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \}$, what is Span $S_3$?
- Let $S_4 = \{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \}$, what is Span $S_4$?

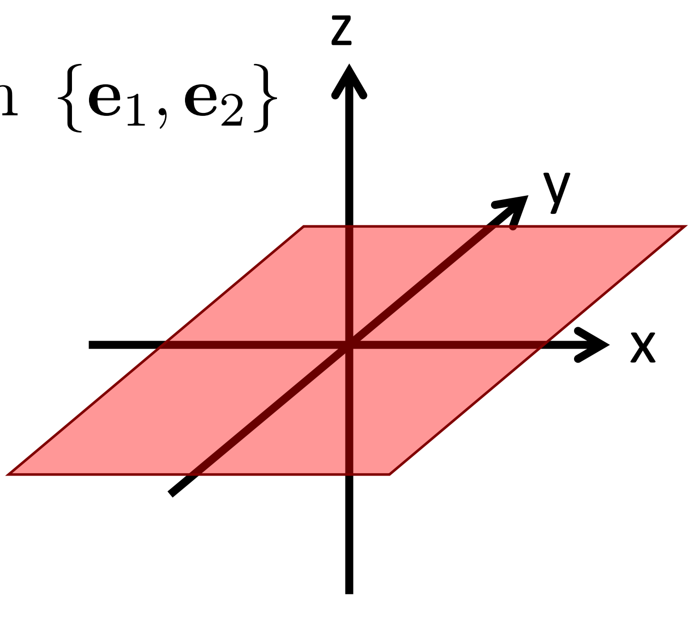


# Span




$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

$$A\mathbf{x} = \mathbf{b}$$

Has solution or not?

The same question

$$x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

Is $b$ the linear combination of columns of $A$?

The same question

$$\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \in Span\left\{ \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} \ldots \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} \right\}$$

Is $b$ in the span of the columns of $A$?

# Summary

$$A\mathbf{x} = \mathbf{b}$$

Does a system of linear equations have solution?

Is $b$ a linear combination of columns of $A$?

Is $b$ in the span of the columns of $A$?

YES ➡ Have solution

NO ➡ No solution