7.3

$S^{\perp}$ is the set of vectors that are orthogonal to every vector in S

$$S^{\perp} = \{v: v \cdot u = 0, \forall u \in S\}$$

Unique

$$u = \underbrace{\boxed{c_1w_1 + c_2w_2 + \cdots + c_kw_k}}_{w}$$
$$\underbrace{\boxed{+b_1z_1 + b_2z_2 + \cdots + b_{n-k}z_{n-k}}}_{z}$$

For any subspace W of R^n

$$dimW + dimW^{\perp} = n$$

Basis: $\{w_1, w_2, \cdots, w_k\}$ Basis: $\{z_1, z_2, \cdots, z_{n-k}\}$

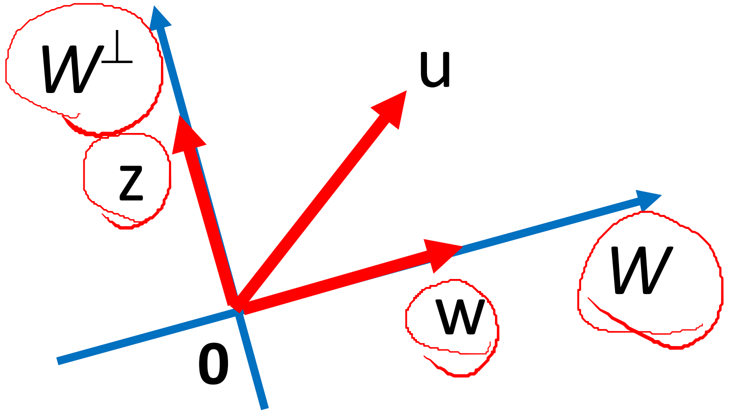
Basis for R^n

For ***every*** vector u,

u = w + z ***(unique)***

w $\in W$

z $\in W^{\perp}$



58. Let W be a subspace of R^n, and let B_1 and B_2 be bases for W and $W^{\perp}$, respectively.

(a) $B_1 \cup B_2$ is a basis for R^n

(b) $dimW + dimW^{\perp} = n$

4-2: 77 Let V and W be nonzero subspaces of R^n such that each vector $\boldsymbol{u}$ in R^n can be uniquely expressed in the form $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$ for some $\boldsymbol{v}$ in V and some $\boldsymbol{w}$ in W.

$V = W^{\perp}$

(a) Prove that $\mathbf{0}$ is the only vector in both $\overset{W^{\perp}}{\cancel{V}}$ and W.

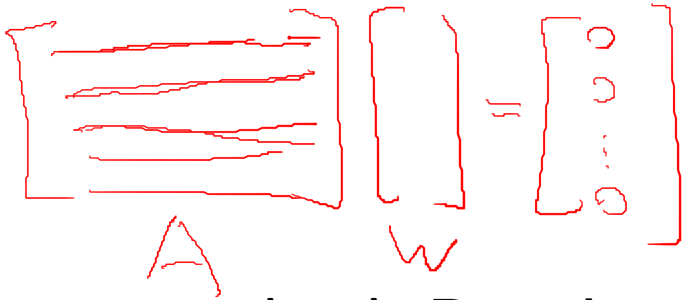
(b) Prove that $dim\overset{W^{\perp}}{\cancel{V}} + dimW = n$

61. Prove the following statements for any matrix A:

(a) $(Row\ A)^{\perp} = Null\ A$

(b) $(Col\ A)^{\perp} = Null\ A^{T}$

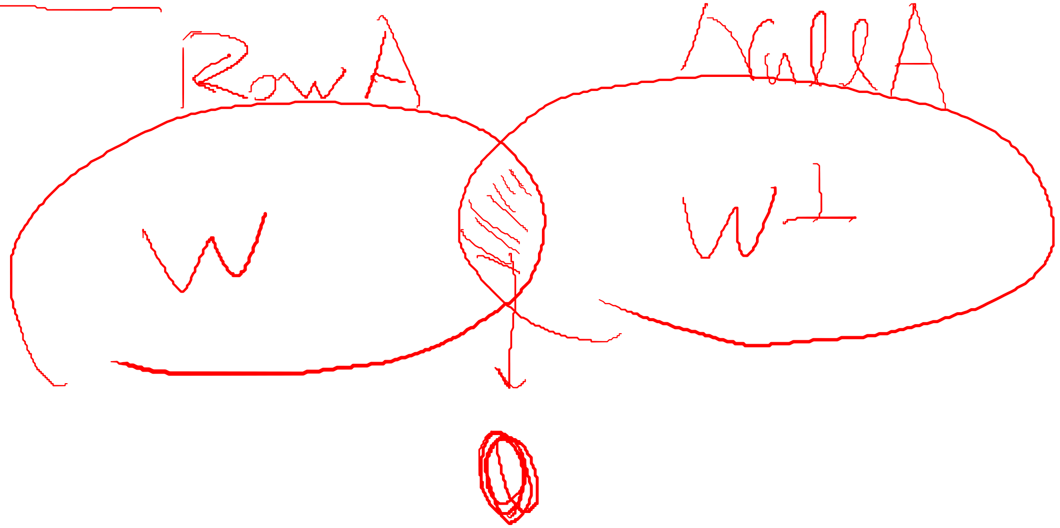
$w \in Null A$



65. Let A be an $n \times n$ matrix. Prove that if $\boldsymbol{v}$ is a vector in both RowA and $NullA$, then $\boldsymbol{v} = 0$.

W

$W^{\perp}$



57. Let S be a nonempty finite subset of R^n, and suppose that $W = Span\ S$. Prove that $W^{\perp} = S^{\perp}$.

$v \in W^{\perp}$ $\quad S \subset W$

$\Rightarrow$ for all $x \in W$, $x \cdot v = 0$

$\Rightarrow$ for all $x \in S$, $x \cdot v = 0$

$\Rightarrow v \in S^{\perp}$

$u \in S^{\perp}$ $\quad \{x_1, x_2 \cdots x_k\}$

$\Rightarrow$ for all $x \in S$, $x \cdot u = 0$

$w = c_1 x_1 + c_2 x_2 + \cdots c_k x_k$

$w \cdot u = c_1 \underbrace{(x_1 \cdot u)}_{=0} + \cdots + c_k \underbrace{(x_k \cdot u)}_{=0}$

$= 0$

$\Rightarrow$ for all $w \in W$, $w \cdot u = 0$

$\Rightarrow u \in W^{\perp}$

60. Prove that for any subspace W of R^n, $(W^{\perp})^{\perp} = W$

$W^{\perp}$

$\to W$

63. Prove that for any nonempty finite subset S of R^n, $(S^{\perp})^{\perp} = Span\ S$

① $W \subseteq (W^{\perp})^{\perp}$

$(W^{\perp})^{\perp}$

W

②

$\dim W^{\perp} + \dim (W^{\perp})^{\perp} = n$

$\dim W + \dim W^{\perp} = n$

$\dim W = k$

$\dim W^{\perp} = n-k$

$\dim (W^{\perp})^{\perp} = ?\ k$

$W = (W^{\perp})^{\perp}$

57. Let S be a nonempty finite subset of R^n, and suppose that $W = Span\ S$. Prove that $W^{\perp} = S^{\perp}$.

64. Use the fact the $(Row\ A)^{\perp} = Null\ A$ for any matrix A to give another proof that $dimW + dimW^{\perp} = n$ for any subspace W of R^n.

Hint: Let A be a $k \times n$ matrix whose rows constitute a basis for W.

$\boxed{W = Row\ A}$

basis $= \{w_1, w_2, \ldots, w_k\}$

$$k\left[\begin{array}{c} \text{———} \; w_1 \; \text{———} \\ \text{———} \; w_2 \; \text{———} \\ \vdots \\ \text{———} \; w_k \; \text{———} \end{array}\right]$$

n

$\dim W + \dim W^{\perp}$

$= \dim(Row\,A) + \dim(Row\,A)^{\perp}$

$= \underbrace{\boxed{\dim(Row\,A)}}_{= rank\,A} + \underbrace{\boxed{\dim(Null\,A)}}_{= n - rank\,A}$

$= n$

orthogonal, ind.

59. Suppose that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ is an orthogonal basis for R^n. For any k, where $1 \le k < n$, define $W = Span\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k\}$. Prove that $\{\boldsymbol{v}_{k+1}, \boldsymbol{v}_{k+2}, \dots, \boldsymbol{v}_n\}$ is an orthogonal basis for $W^{\perp}$.

$= U$

orthogonal.

$\dim W + \dim W^{\perp} = n$

$k \qquad n-k$

⓪ $U \subset W^{\perp}$

① U is ind. ✓

② no. of U $(n-k)$

$= \dim W^{\perp}$ $(n-k)$

✓

if $v \in U$

$v \cdot w = 0$ all $w \in W$

$w = c_1 v_1 + c_2 v_2 + \dots c_k v_k$

67. Let W be a subspace of R^n.

(a) Prove that $(P_W)^2 = P_W$

(b) Prove that $(P_W)^T = P_W$

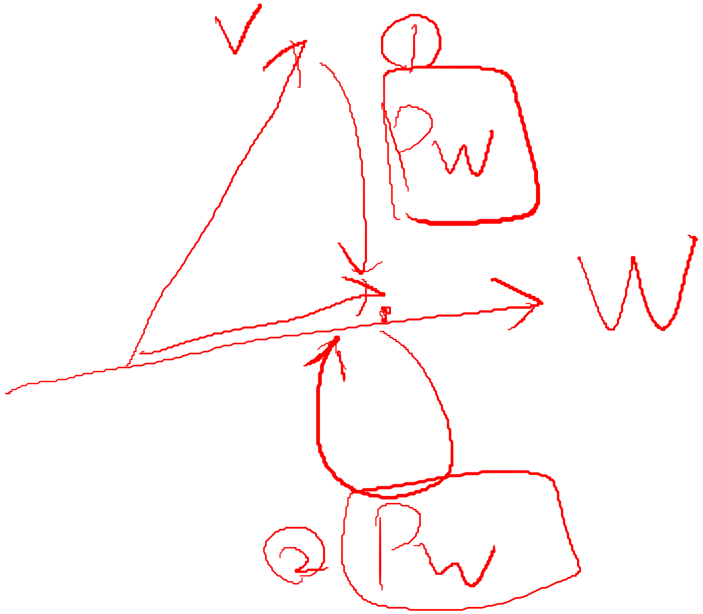
$$\boxed{P_W = C(C^TC)^{-1}C^T}$$

C: W's basis

$(P_W)^2 =$

(a) $C(C^TC)^{-1}\cancel{C^TC(C^TC)^{-1}}C^T$

$= C(C^TC)^{-1}C^T = P_W$



67. Let W be a subspace of R^n.

(a) Prove that $(P_W)^2 = P_W$

(b) Prove that $(P_W)^T = P_W$

$$P_W = C(C^TC)^{-1}C^T$$

$$(P_W)^T = \left(C(C^TC)^{-1}C^T\right)^T$$

$$= (C^T)^T((C^TC)^{-1})^T C^T$$

$$= C\left((C^TC)^T\right)^{-1}C^T$$

$$= C(C^TC)^{-1}C^T = P_W$$

$$(C^TC)^T$$

$$= C^T(C^T)^T$$

$$= \underline{C^TC}$$

72. Let W be a subspace of R^n. Prove that $P_W P_{W^\perp} = P_{W^\perp} P_W = O$

$P_W = C(C^TC)^{-1}C^T$ C: W's basis

$P_{W^\perp} = B(B^TB)^{-1}B^T$ B: $W^\perp$'s basis

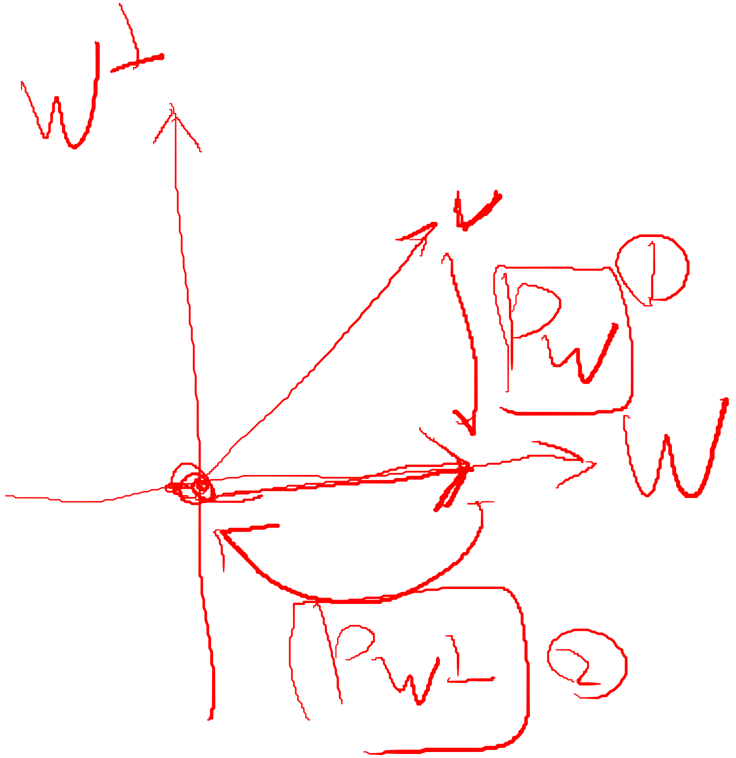
$P_{W^\perp} P_W = O$

$= B(B^TB)^{-1}\boxed{B^TC}(C^TC)^{-1}C^T = O$

$W^\perp$'s basis

W's basis

72. Let W be a subspace of R^n. Prove that $P_W P_{W^\perp} = P_{W^\perp} P_W = O$



$u \in R^n$

u is any vector

$u = w + z$

$w \in W$

$z \in W^\perp$

$$P_{W^\perp} P_W \overset{(w+z)}{u}$$

$$= P_{W^\perp} P_W w + P_{W^\perp} \underbrace{P_W z}_{0}$$

$$= P_{W^\perp} w + 0$$

$$= 0$$

$$P_{W^\perp} P_W = 0$$

73. Let W be a subspace of R^n. Prove that $P_W + P_{W^\perp} = I_n$.

$u = w + z$

$w \in W,\ z \in W^\perp$

$$P_W + P_{W^\perp}$$
$$= C(C^TC)^{-1}C^T + B(B^TB)^{-1}B^T$$
$$\cdots$$
$$= I$$

$$\underbrace{(P_W + P_{W^\perp})}_{I}(u) \qquad u \in R^n$$
$$= (P_W + P_{W^\perp})(w + z)$$
$$= \underbrace{P_W w}_{=w} + \underbrace{P_W z}_{=0} + \underbrace{P_{W^\perp} w}_{=0} + \underbrace{P_{W^\perp} z}_{=z}$$
$$= w + z = u$$