Linear Combination

Linear Combination

- Given a vector set $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\}$
- The linear combination of the vectors in the set
  - $\boldsymbol{v} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}$
  - $c_1, c_2, \cdots, c_k$ are scalars (coefficients of linear combination)

vector set: $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}$

coefficients: $\{-3,4,1\}$

$$-3\begin{bmatrix}1\\1\end{bmatrix} + 4\begin{bmatrix}1\\3\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}2\\8\end{bmatrix}$$

其實就是 weighted sum 啦 ☺

Column Aspect

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

$\boldsymbol{a_1}$ $\boldsymbol{a_2}$ $\boldsymbol{a_n}$

$$A = [\boldsymbol{a_1} \quad \boldsymbol{a_2} \quad \cdots \quad \boldsymbol{a_n}]$$

Vector set

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

coefficients

$$A\mathbf{x} = x_1\boldsymbol{a_1} + x_2\boldsymbol{a_2} + \cdots + x_n\boldsymbol{a_n}$$

Linear Combination

System of Linear Equations v.s. Linear Combination

$$A\mathbf{x} = \mathbf{b}$$

(A system of linear equations)

Non empty solution set?

Has solution or not?

Consistent?

The Same question

Column Aspect

$$A\mathbf{x} = b$$

$$= x_1\boldsymbol{a_1} + x_2\boldsymbol{a_2} + \cdots + x_n\boldsymbol{a_n}$$

linear combination of columns of A

Is $\boldsymbol{b}$ the linear combination of columns of A?

Example 1

$$3x_1 + 6x_2 = 3$$
$$2x_1 + 4x_2 = 4$$

Has solution or not?

$$A\mathbf{x} = \mathbf{b}$$

$A = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $b = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

Is b the linear combination of columns of A?

$\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ $\left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \end{bmatrix} \right\}$

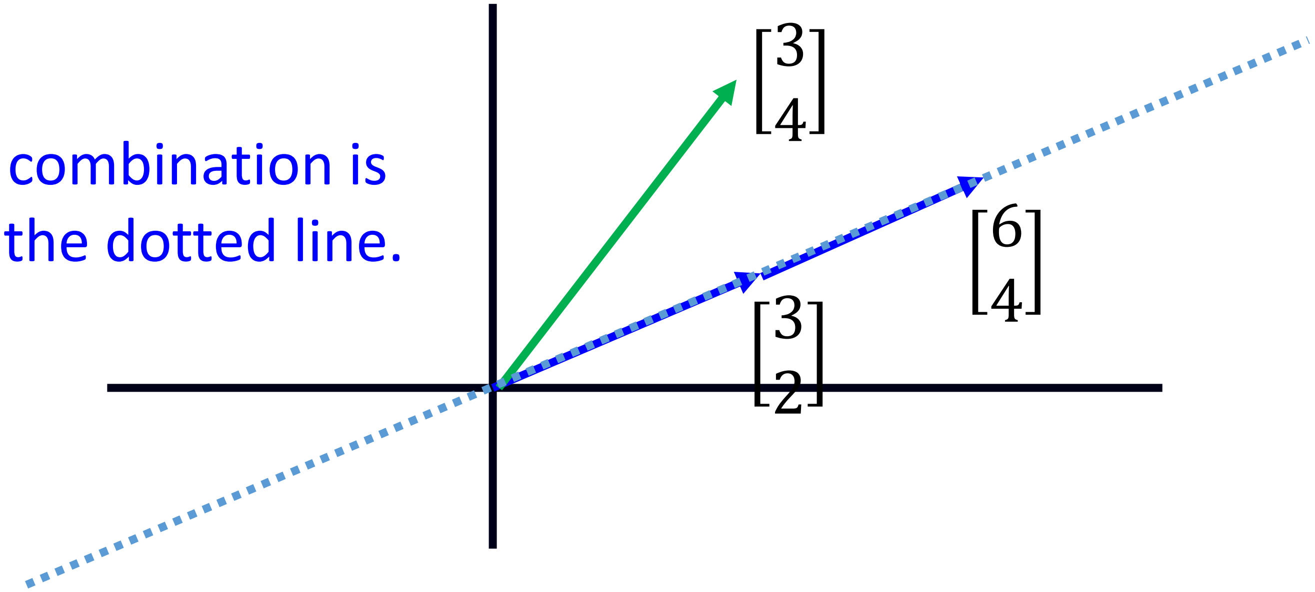
Example 1

$$3x_1 + 6x_2 = 3$$
$$2x_1 + 4x_2 = 4$$

Has solution or not?

- Vector set: $\left\{\begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}6\\4\end{bmatrix}\right\}$
- Is $\begin{bmatrix}3\\4\end{bmatrix}$ a linear combination of $\left\{\begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}6\\4\end{bmatrix}\right\}$? **NO**

The linear combination is always on the dotted line.



Example 2

$$2x_1 + 3x_2 = 4$$
$$3x_1 + 1x_2 = -1$$

Has solution or not?

$$A\mathbf{x} = \mathbf{b}$$

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

Is b the linear combination of columns of A?

$$\begin{bmatrix} 4 \\ -1 \end{bmatrix} \qquad \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$$

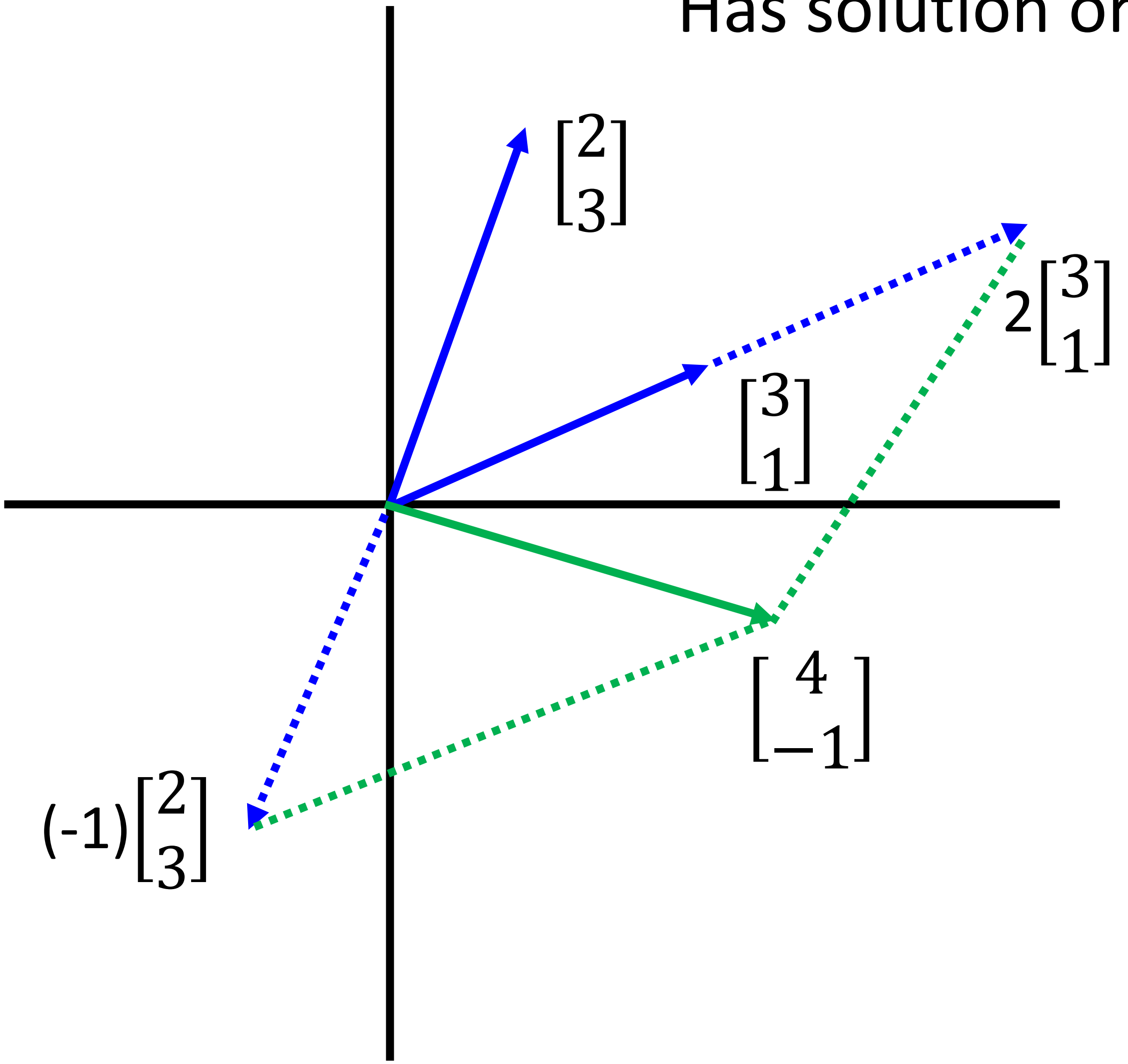
Example 2

$$2x_1 + 3x_2 = 4$$
$$3x_1 + 1x_2 = -1$$

Has solution or not?

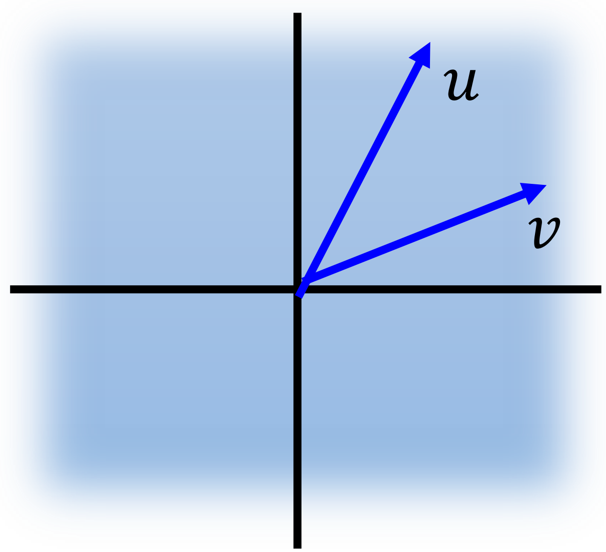
$\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$

$\begin{bmatrix} 4 \\ -1 \end{bmatrix}$



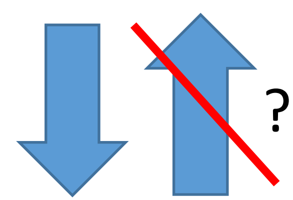
Example 2

- If **u** and **v** are any nonparallel vectors in $\mathcal{R}^2$, then every vector in $\mathcal{R}^2$ is a linear combination of **u** and **v**
  - Nonparallel: **u** and **v** are nonzero vectors, and **u** ≠ c**v**.



$$
\begin{aligned}
u_1 x_1 + v_1 x_2 &= b_1 \\
u_2 x_1 + v_2 x_2 &= b_2
\end{aligned}
$$

u and **v** are not parallel



Has solution

- If **u, v** and **w** are any nonparallel vectors in $\mathcal{R}^3$, then every vector in $\mathcal{R}^3$ is a linear combination of **u, v** and **w**? **NO**

Example 3

$$2x_1 + 6x_2 = -4$$
$$1x_1 + 3x_2 = -2$$

Has solution or not?

$$A\mathbf{x} = \mathbf{b}$$

$A = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $b = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$

Is b the linear combination of columns of A?

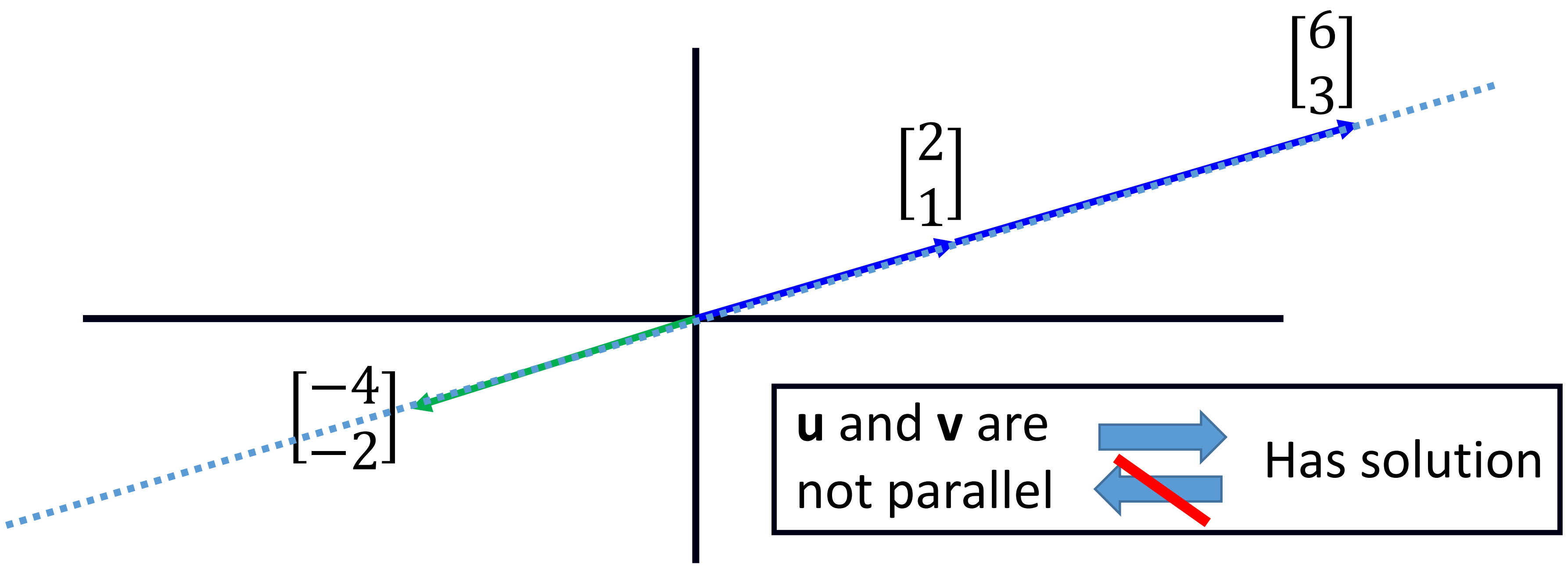
$\begin{bmatrix} -4 \\ -2 \end{bmatrix}$ $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \end{bmatrix} \right\}$

Example 3

$$\color{blue}{2}\color{red}{x_1} + \color{blue}{6}\color{red}{x_2} = \color{green}{-4}$$
$$\color{blue}{1}\color{red}{x_1} + \color{blue}{3}\color{red}{x_2} = \color{green}{-2}$$

Has solution or not?

- Vector set: $\left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}6\\3\end{bmatrix}\right\}$
- Is $\begin{bmatrix}-4\\-2\end{bmatrix}$ a linear combination of $\left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}6\\3\end{bmatrix}\right\}$? Yes



Summary

$$A\mathbf{x} = \mathbf{b}$$

$A: m \times n \quad \boldsymbol{x} \in R^n \quad \boldsymbol{b} \in R^m$

Is ***b*** a ***linear combination*** of columns of A?

Is ***b*** in the ***span*** of the columns of A?

NO

No solution

YES

The columns of A are ***independent***.

Rank A = n

Nullity A = 0

Unique solution

The columns of A are ***dependent***.

Rank A < n

Nullity A > 0

Infinite solution