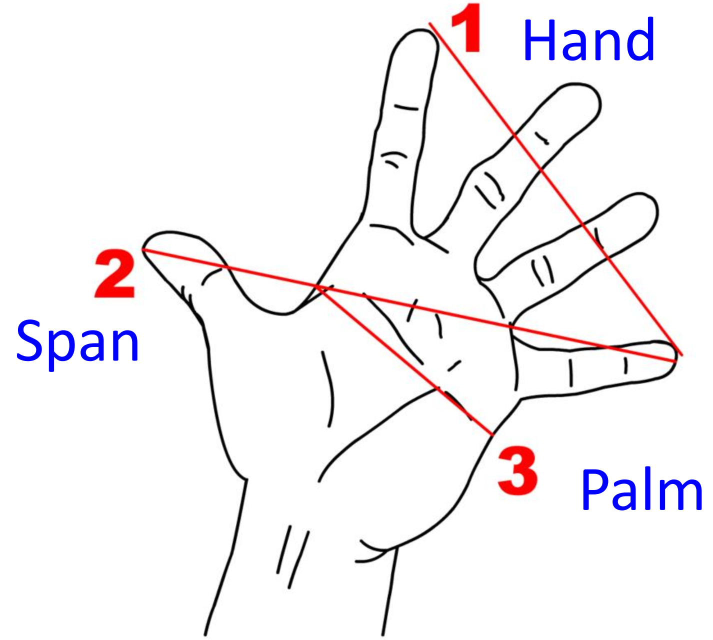
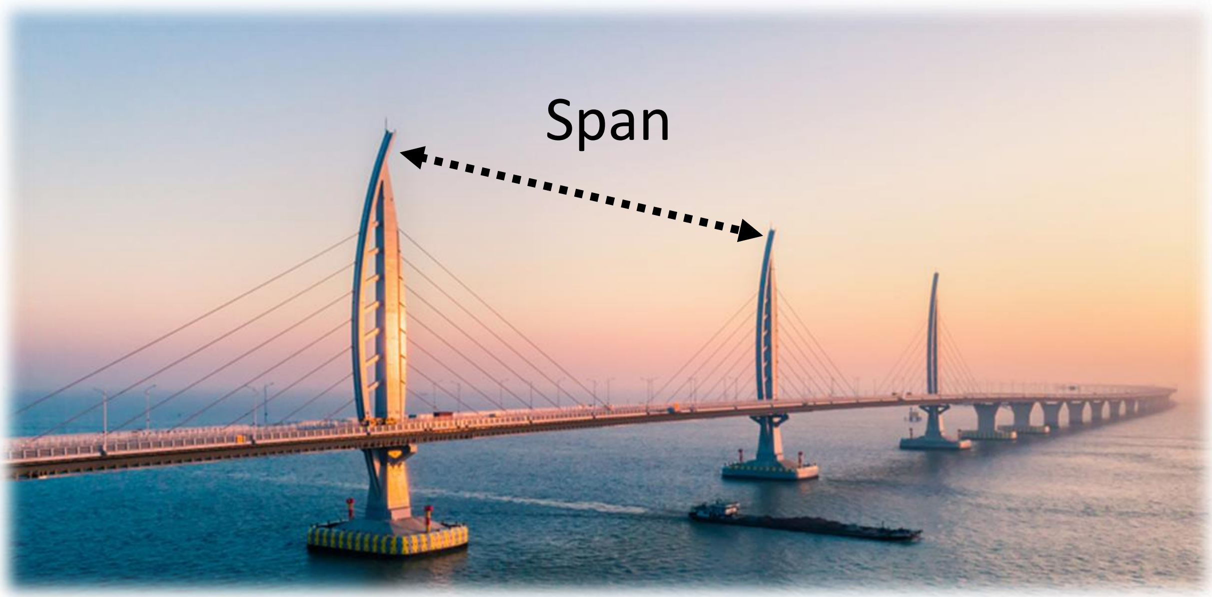
Span





Span

- A vector set $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_k\}$
- $Span\ S$ is the vector set of all linear combinations of $\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_k$

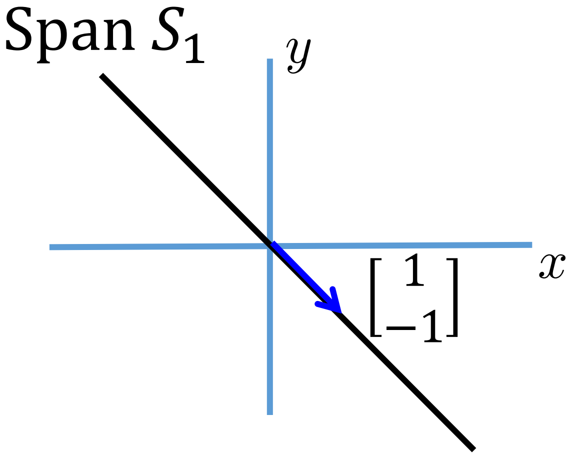
$$Span\ S = \{c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k |\ for\ all\ c_1, c_2, \cdots, c_k\}$$

- Vector set $V = Span\ S$
  - "S is a generating set for V" or "S generates V"
  - One way to describe a vector set with infinite elements

Span

$$c_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Let $S_0 = \{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \}$, what is Span S_0?
  - Ans: $\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \}$ (only one member)



- Let $S_1 = \{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \}$, what is Span S_1?
  - Span $S_1 = \{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \dots\dots \}$
  - If S contains a non zero vector, then Span S has infinitely many vectors

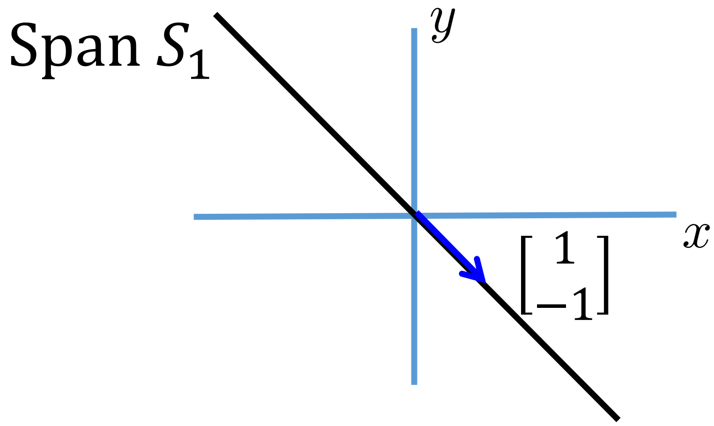
Span

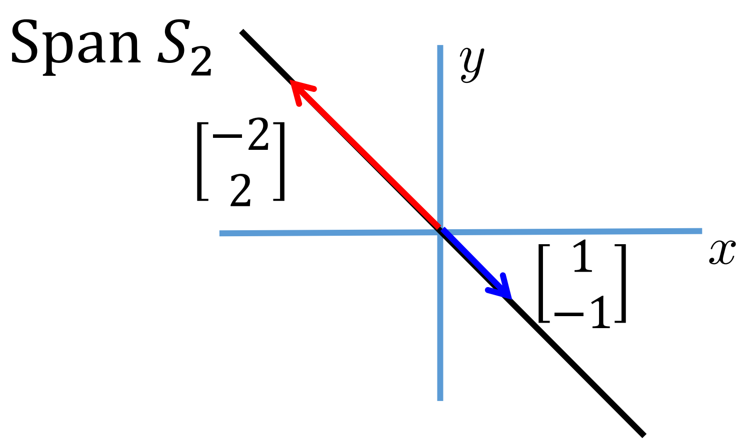
Let $S_1 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

$\text{Span } S_1 = ?$

Let $S_2 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right\}$

what is $\text{Span } S_2$?



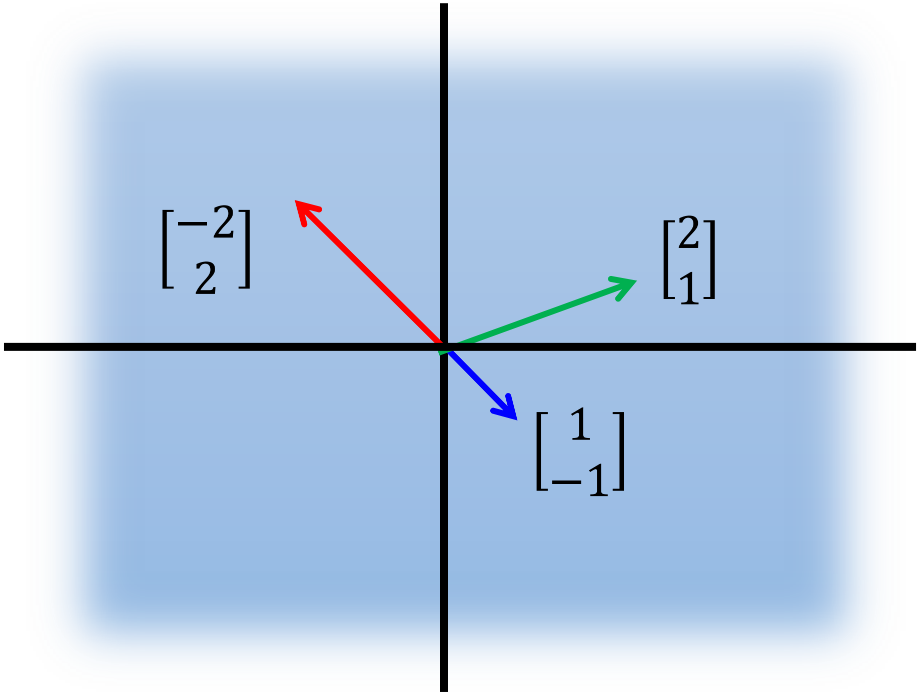


$$\text{Span } S_1 = \text{Span } S_2$$

(Different number of vectors can generate the same space.)

Span

- Let $S_3 = \{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\}$, what is Span S_3?



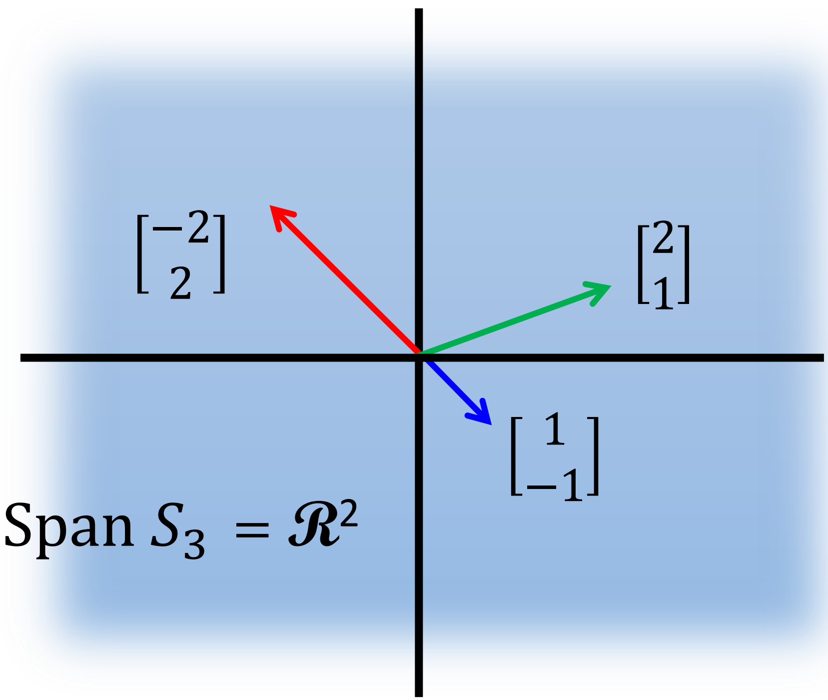
Every vector in $\mathcal{R}^2$ is their linear combination

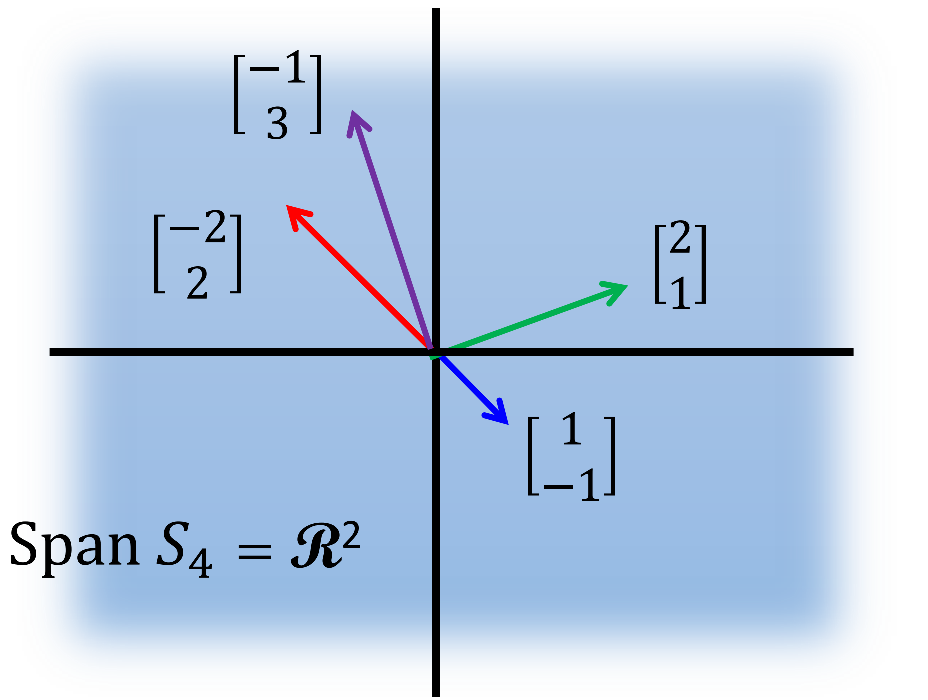
$$\text{Span } S_3 = \mathcal{R}^2$$

Span

Let $S_3 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$
what is Span S_3 =?

Let $S_4 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$
what is Span S_4 =?





$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

$$A\mathbf{x} = \mathbf{b}$$

Has solution or not?

The same question

$$x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

Is b the linear combination of columns of A?

The same question

$$\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \in Span \left\{ \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} \ldots \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} \right\}$$

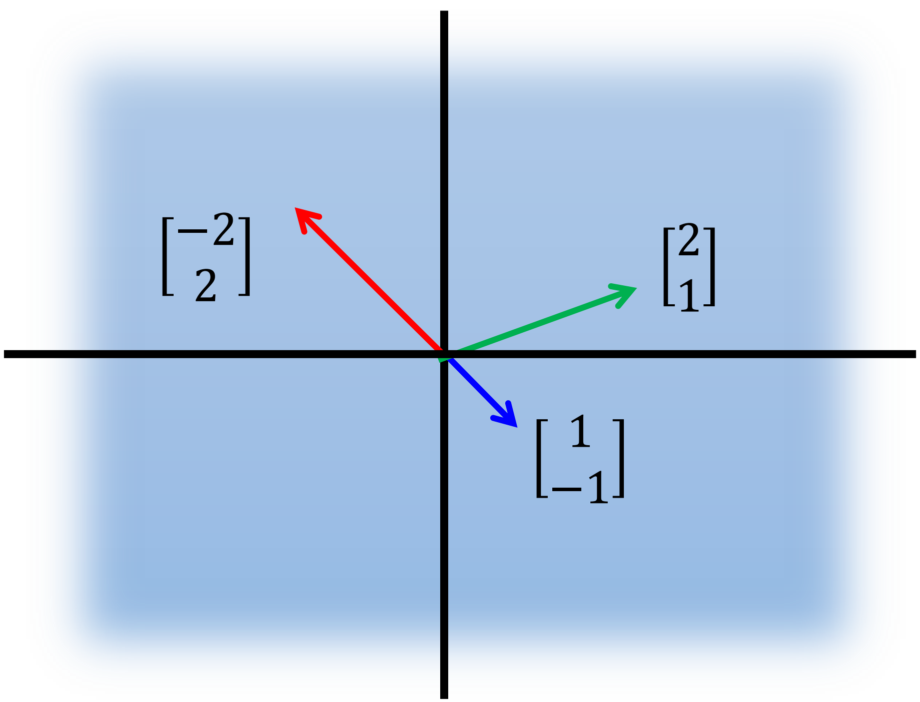
Is b in the span of the columns of A?

Span

$A\mathbf{x} = \mathbf{b}$

$\begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ Every **b** has solution

- Let $S_3 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$, what is Span S_3?



Every vector in $\mathcal{R}^2$ is their linear combination

$\text{Span } S_3 = \mathcal{R}^2$

Summary

$$A\mathbf{x} = \mathbf{b}$$

$A: m \times n$ $\quad$ $\boldsymbol{x} \in R^n$ $\quad$ $\boldsymbol{b} \in R^m$

Is b a ***linear combination*** of columns of A?

Is b in the ***span*** of the columns of A?

NO

No solution

YES

- The columns of A are ***independent***.
  - ***Rank*** A = n
  - ***Nullity*** A = 0
  - Unique solution
- The columns of A are ***dependent***.
  - ***Rank*** A < n
  - ***Nullity*** A > 0
  - Infinite solution