Orthogonal Matrices & Symmetric Matrices

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Outline

Orthogonal Matrices

- Reference: Chapter 7.5

Symmetric Matrices

- Reference: Chapter 7.6

Orthogonal Matrix

- An nxn matrix Q is called an **orthogonal matrix** if the columns of Q are orthonormal.
- Orthogonal operator: standard matrix is an orthogonal matrix.

$$A_\theta = \underset{\text{orthogonal}}{\overset{\begin{matrix}\text{unit} & \quad\text{unit}\end{matrix}}{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}}$$ is an orthogonal matrix.

Norm-preserving

- A linear operator is norm-preserving if

$$\|T(u)\| = \|u\| \qquad \text{For all u}$$

Example: linear operator *T* on R^2 that rotates a vector by θ.

$\Rightarrow$ Is *T* norm-preserving?

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Example: linear operator *T* is reflection

$\Rightarrow$ Is *T* norm-preserving?

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Norm-preserving

- A linear operator is norm-preserving if

$$\|T(u)\| = \|u\| \qquad \text{For all u}$$

Example: linear operator *T* is projection
$\Rightarrow$ Is *T* norm-preserving?

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Example: linear operator *U* on R^n that has an eigenvalue $\lambda \neq \pm 1$.

Norm-preserving

- Necessary conditions:

Norm-preserving $\rightarrow$ Orthogonal Matrix

Norm-preserving $\xleftarrow[???]{}$ Orthogonal Matrix

Linear operator Q is norm-preserving

$\Rightarrow$ $||\mathbf{q}_j|| = 1$ $\qquad$ $||\mathbf{q}_j|| = ||Q\mathbf{e}_j|| = ||\mathbf{e}_j||$

$\Rightarrow$ $\mathbf{q}_i$ and $\mathbf{q}_j$ are orthogonal $\qquad$ 畢式定理

$$||\mathbf{q}_i + \mathbf{q}_j||^2 = ||Q\mathbf{e}_i + Q\mathbf{e}_j||^2 = ||Q(\mathbf{e}_i + \mathbf{e}_j)||^2 = ||\mathbf{e}_i + \mathbf{e}_j||^2 = 2 = ||\mathbf{q}_i||^2 + ||\mathbf{q}_j||^2$$

Orthogonal Matrix

Those properties are used to check orthogonal matrix.

- Q is an orthogonal matrix
- $QQ^T = I_n$
- Q is invertible, and $Q^{-1} = Q^T$ — Simple inverse
- $Qu \cdot Qv = u \cdot v$ for any u and v — Q preserves dot projects
- $\|Qu\| = \|u\|$ for any u — Q preserves norms

Norm-preserving $\rightleftarrows$ Orthogonal Matrix

Orthogonal Matrix

1 • Q is an orthogonal matrix
• $Q^TQ = I_n$
2 • Q is invertible, and $Q^{-1} = Q^T$
• $Qu \cdot Qv = u \cdot v$ for any u and v
3 • $\|Qu\| = \|u\|$ for any u

1

$$q_i \cdot q_j = {q_i}^T q_j \begin{cases} i \neq j: & 0 \\ i = j: & 1 \end{cases}$$

Q^TQ

i-j entry is ${q_i}^T q_j$

2 $Qu \cdot Qv = (Qu)^T Qv = u^T Q^T Qv = u^T Iv = u^T v = u \cdot v$

3 $Qu \cdot Qv = u \cdot v \Rightarrow Qu \cdot Qu = u \cdot u$

$\Rightarrow \|Qu\|^2 = \|u\|^2 \Rightarrow \|Qu\| = \|u\|$

Orthogonal Matrix

- Let P and Q be n x n orthogonal matrices
  - $detQ = \pm 1$
  - PQ is an orthogonal matrix
  - Q^{-1} is an orthogonal matrix
  - Q^{T} is an orthogonal matrix

Check by $(PQ)^{-1} = (PQ)^{T}$

Check by $(Q^{-1})^{-1} = (Q^{-1})^{T}$

Proof

(C) $(Q^{-1})^{-1} = (Q^{T})^{-1} = (Q^{-1})^{T}$

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}$$

Rows and columns

Orthogonal Operator

- Applying the properties of orthogonal matrices on orthogonal operators
- T is an orthogonal operator
  - $T(u) \cdot T(v) = u \cdot v$ for all u and v
  - $\|T(u)\| = \|u\|$ for all u

Preserves dot product

Preserves norms

- T and U are orthogonal operators, then TU and T^{-1} are orthogonal operators.

Example: Find an orthogonal operator T on R^3 such that

$$T\left(\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Norm-preserving

$$v = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \qquad Av = e_2 \qquad v = A^{-1}e_2$$

Find A^{-1} first

Because $A^{-1} = A^T$

$$A^{-1} = \begin{bmatrix} * & 1/\sqrt{2} & * \\ * & 0 & * \\ * & 1/\sqrt{2} & * \end{bmatrix}$$

Also orthogonal

$$\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

$$A = (A^{-1})^T$$

$$1/\sqrt{2}\, x_1 + 0x_2 + 1/\sqrt{2}\, x_3 = 0$$

Conclusion

- Orthogonal Matrix (Operator)
  - Columns and rows are orthogonal unit vectors
  - Preserving norms, dot products
  - Its inverse is equal its transpose

Outline

Orthogonal Matrices

- Reference: Chapter 7.5

Symmetric Matrices

- Reference: Chapter 7.6

Eigenvalues are real

- The eigenvalues for symmetric matrices are always real.

Consider 2 x 2 symmetric matrices

How about more general cases?

$$A = A^T = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \in R^{2\times 2}$$

$$det(A - tI_2) = t^2 \underline{- (a + c)}t + \underline{ac - b^2}$$

$$\text{Since } \underline{(a+c)}^2 - 4\underline{(ac-b^2)} = (a-c)^2 + 4b^2 \geq 0$$

The symmetric matrices always have real eigenvalues.

Eigenvalues are real

Symmetric matrix A always has eigenvalue.

A symmetric matrix A has an eigenvalue λ $\quad\quad \lambda = a + bi$

$$\mathrm{Av} = \lambda v \rightarrow \overline{Av} = \overline{\lambda v} \rightarrow \bar{A}\bar{v} = \bar{\lambda}\bar{v} \xrightarrow[\bar{A} = A]{} A\bar{v} = \bar{\lambda}\bar{v}$$

$$(\bar{v})^{\mathrm{T}} A v \rightarrow (\bar{v})^{\mathrm{T}} \lambda v = \boxed{\lambda (\bar{v})^{\mathrm{T}}\ v}$$

$$(\bar{v})^{\mathrm{T}} A v \rightarrow (\bar{v})^{\mathrm{T}} A^{\mathrm{T}} v = (A\bar{v})^{\mathrm{T}}\ v$$

$$= \left(\bar{\lambda}\bar{v}\right)^{\mathrm{T}}\ v = \boxed{\bar{\lambda}(\bar{v})^{\mathrm{T}}\ v}$$

$$v = \begin{bmatrix} a_1 + b_1 i \\ a_2 + b_2 i \\ \vdots \end{bmatrix} \neq \mathbf{0}$$

$$\bar{v} = \begin{bmatrix} a_1 - b_1 i \\ a_2 - b_2 i \\ \vdots \end{bmatrix}$$

$$(\bar{v})^{\mathrm{T}} v = ? \quad > 0 \rightarrow \lambda = \bar{\lambda} \rightarrow b = 0$$

$$= (a_1)^2 + (b_1)^2 + (a_2)^2 + (b_2)^2 + \cdots$$

Orthogonal Eigenvectors

$$det(A - tI_n)$$ Factorization

A is symmetric

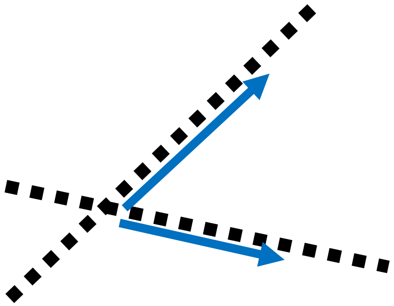
$$= (t - \lambda_1)^{\underline{m_1}}(t - \lambda_2)^{\underline{m_2}} \ldots (t - \lambda_k)^{\underline{m_k}}(\ldots \ldots)$$

Eigenvalue: λ_1 λ_2 …… λ_k

Eigenspace: $d_1 \leq m_1$ $d_2 \leq m_2$ …… $d_k \leq m_k$

(dimension)

Independent → orthogonal



Orthogonal Eigenvectors

- A is symmetric.
- If u and v are eigenvectors corresponding to eigenvalues λ and μ ($\lambda \neq \mu$)

$\Rightarrow$ u and v are orthogonal.

$$A\mathbf{u} \cdot \mathbf{v} = \lambda \mathbf{u} \cdot \mathbf{v}$$

$$A\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot A^T \mathbf{v}$$

$$= \mathbf{u} \cdot \mu \mathbf{v}$$

Diagonalization

$\mathrm{A} = P\mathrm{D}\mathrm{P}^{T}$

A is symmetric $\rightleftarrows$ $\mathrm{P}^{T}AP = \mathrm{D}$

P is an orthogonal matrix

D is a diagonal matrix

$\Leftarrow$: simple

$\mathrm{P}^{T}AP = \mathrm{D}$ $\Rightarrow$ $\mathrm{P}^{-1}AP = \mathrm{D}$

$\Rightarrow$ $\mathrm{A} = P\mathrm{D}\mathrm{P}^{-1}$ Diagonalization

$\mathrm{A} = P\mathrm{D}\mathrm{P}^{T}$

P consists of eigenvectors , D are eigenvalues

Diagonalization

A is symmetric $\rightarrow$ $P^T AP = D$

$A: n \times n$

- A has eigenvalue λ $\quad A\boldsymbol{u}_1 = \lambda \boldsymbol{u}_1 \quad$ $\boldsymbol{u}_1$ is unit vector
- Find an orthonormal basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_n\} = B$

eigenvector $\quad$ don't care

by the Extension Theorem and Gram-Schmidt Process

$$B^T AB = ? \quad (B^T AB)^T = B^T A^T (B^T)^T = B^T AB$$

symmetric

$$B^T AB\boldsymbol{e}_1 = B^T A\boldsymbol{u}_1 = B^T \lambda \boldsymbol{u}_1 = \lambda B^T \boldsymbol{u}_1$$

$$= \lambda \begin{bmatrix} {\boldsymbol{u}_1}^T \\ {\boldsymbol{u}_2}^T \\ \vdots \end{bmatrix} \boldsymbol{u}_1 = \lambda \begin{bmatrix} 1 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} \lambda \\ 0 \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} \lambda & \mathbf{0} \\ \mathbf{0} & A' \end{bmatrix}$$

symmetric

Diagonalization

A is symmetric $\Rightarrow$ $P^TAP = D$

$A: n\times n$

ortho

$$B^TAB = \begin{bmatrix}\lambda & 0\\ 0 & A'\end{bmatrix}$$

sym

symmetric

$$B'^TA'B' = \boxed{\begin{bmatrix}\lambda' & 0\\ 0 & A''\end{bmatrix}}$$

symmetric

$$C = \begin{bmatrix}1 & 0\\ 0 & B'\end{bmatrix}$$

B' ortho

ortho

$C^TB^TABC = ?$

$$\begin{bmatrix}1 & 0\\ 0 & B'^T\end{bmatrix}\begin{bmatrix}\lambda & 0\\ 0 & A'\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & B'\end{bmatrix} = \begin{bmatrix}\lambda & 0\\ 0 & \boxed{B'^TA'B'}\end{bmatrix}$$

Diagonalization

A is symmetric $\rightarrow$ $P^TAP = D$

$A: n\times n$

$C^TB^TABC = ?$

$$\begin{bmatrix} 1 & 0 \\ 0 & B'^T \end{bmatrix}\begin{bmatrix} \lambda & 0 \\ 0 & A' \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & B' \end{bmatrix}$$

$$B'^TA'B' = \begin{bmatrix} \lambda' & 0 \\ 0 & A'' \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & 0 \\ 0 & B'^TA'B' \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda' & 0 \\ 0 & 0 & A'' \end{bmatrix}$$

$$\dots \overbrace{C^TB^T}^{\text{ortho}} A \overbrace{BC}^{\text{ortho}} \dots$$

$$= D$$

Diagonalization

- Example

$$A = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$$

$A = PDP^{-1}$ $\Rightarrow$ $A = PDP^{T}$, $P^{T}AP = D$

A has eigenvalues $\lambda_1 = 6$ and $\lambda_2 = 1$,

with corresponding eigenspaces $E_1 = \text{Span}\{[\,-1\ \ 2\,]^T\}$ and $E_2 = \text{Span}\{[\,2\ \ 1\,]^T\}$

orthogonal

$\Rightarrow B_1 = \{[\,-1\ \ 2\,]^T/\sqrt{5}\}$ and $B_2 = \{[\,2\ \ 1\,]^T/\sqrt{5}\}$

$$P = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}.$$

Example of Diagonalization of Symmetric Matrix

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

$\mathrm{A} = PDP^{-1}$ ➡ $\mathrm{A} = PDP^{T}$

P is an orthogonal matrix

$\lambda_1 = 2$

independent

Eigenspace: $Span\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$ $\xrightarrow[\text{normalization}]{\text{Gram-Schmidt}}$ $Span\left\{\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}\right\}$

$\lambda_2 = 8$

Not orthogonal

Eigenspace: $Span\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$ $\xrightarrow[\text{normalization}]{}$ $Span\left\{\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}\right\}$

$$P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Diagonalization

P is an orthogonal matrix

A is symmetric

$$\mathrm{P}^{T}\mathrm{A}P = \mathrm{D}$$

$$\mathrm{A} = P\mathrm{D}\mathrm{P}^{T}$$

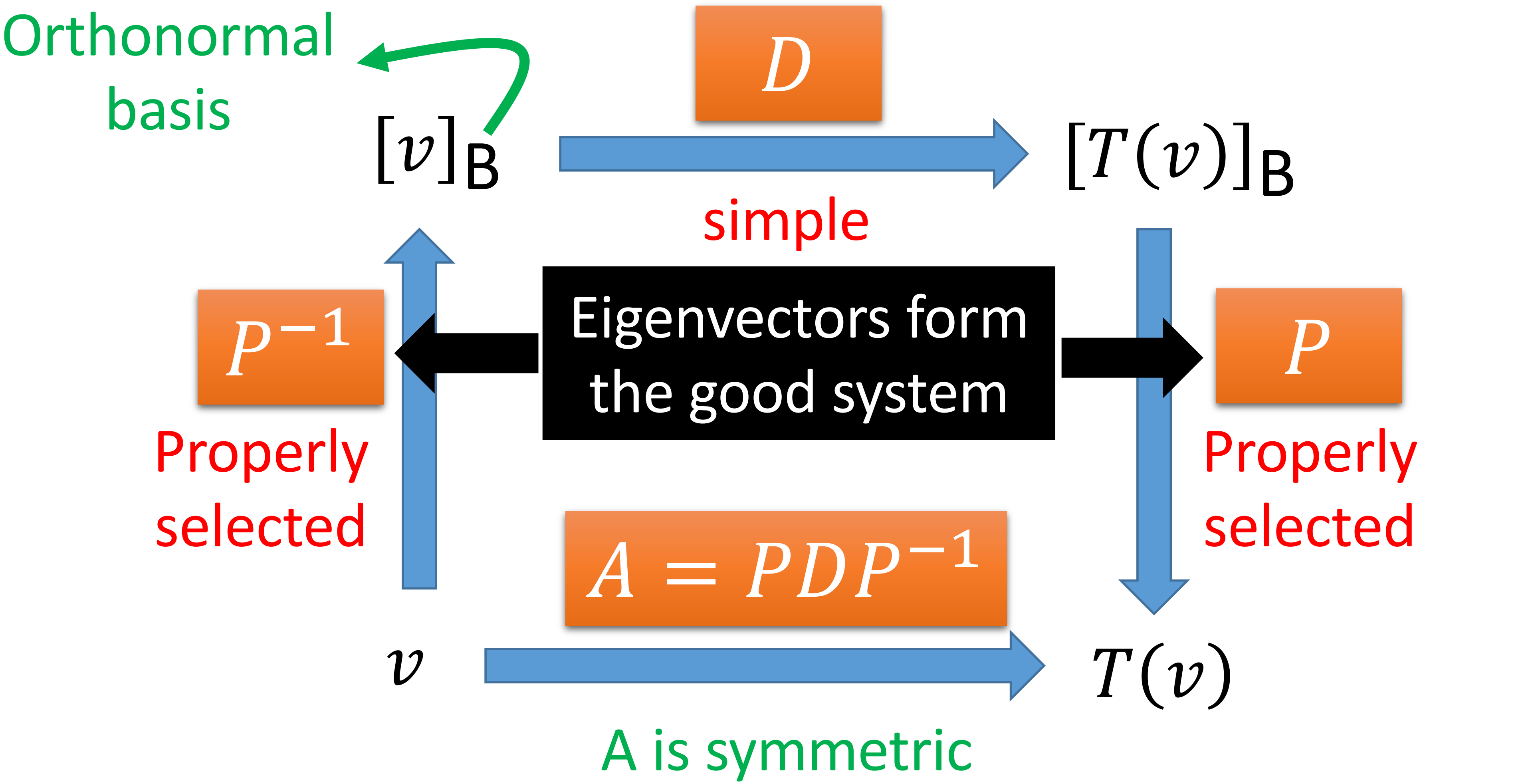
P consists of eigenvectors , D are eigenvalues

Finding an orthonormal basis consisting of eigenvectors of A

Diagonalization of Symmetric Matrix

$$u = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

$$u \cdot v_1 \quad u \cdot v_2 \quad u \cdot v_k$$



Spectral Decomposition

Orthonormal basis

$A = PDP^T$ Let $P = [\, \mathbf{u}_1 \;\; \mathbf{u}_2 \;\; \cdots \;\; \mathbf{u}_n \,]$ and $D = \text{diag}[\, \lambda_1 \;\; \lambda_2 \;\; \cdots \;\; \lambda_n \,]$.

$= P[\, \lambda_1\mathbf{e}_1 \;\; \lambda_2\mathbf{e}_2 \;\; \cdots \;\; \lambda_n\mathbf{e}_n \,]P^T$

$= [\, \lambda_1 P\mathbf{e}_1 \;\; \lambda_2 P\mathbf{e}_2 \;\; \cdots \;\; \lambda_n P\mathbf{e}_n \,]P^T$

$$= [\, \lambda_1\mathbf{u}_1 \;\; \lambda_2\mathbf{u}_2 \;\; \cdots \;\; \lambda_n\mathbf{u}_n \,] \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix}$$

nx1 1xn

P_1 P_2 P_n

$= \lambda_1 \mathrm{P}_1 + \lambda_2 \mathrm{P}_2 + \cdots + \lambda_n \mathrm{P}_n$ P_i are symmetric

Spectral Decomposition

Orthonormal basis

$A = PDP^T$ Let $P = [\ \mathbf{u}_1\ \ \mathbf{u}_2\ \ \cdots\ \ \mathbf{u}_n\ ]$ and $D = \text{diag}[\ \lambda_1\ \ \lambda_2\ \ \cdots\ \ \lambda_n\ ]$.

$$= \lambda_1 \mathrm{P}_1 + \lambda_2 \mathrm{P}_2 + \cdots + \lambda_n \mathrm{P}_n$$

$$\text{rank } P_i = \text{rank } \mathbf{u}_i \mathbf{u}_i^T = 1.$$

$$P_i P_i = \mathbf{u}_i \underline{\mathbf{u}_i^T \mathbf{u}_i} \mathbf{u}_i^T = \mathbf{u}_i \mathbf{u}_i^T$$

$$P_i P_j = \mathbf{u}_i \underline{\mathbf{u}_i^T \mathbf{u}_j} \mathbf{u}_j^T = O$$

$$P_i \mathbf{u}_i = \boldsymbol{u}_i$$

$$P_i \mathbf{u}_j = \mathbf{0}$$

Spectral Decomposition

- Example

$$A = \begin{bmatrix} 3 & -4 \\ -4 & -3 \end{bmatrix}$$ Find spectrum decomposition.

Eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -5$.

$$P_1 = u_1 u_1^T$$

An orthonormal basis consisting of eigenvectors of *A* is

$$B = \left\{ \underbrace{\begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}}_{u_1}, \underbrace{\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}}_{u_2} \right\}$$

$$P_2 = u_2 u_2^T$$

$$A = \lambda_1 P_1 + \lambda_2 P_2$$

Conclusion

- Any symmetric matrix
  - has only real eigenvalues
  - has orthogonal eigenvectors.
  - is always diagonalizable

A is symmetric $\Leftrightarrow$ $P^T AP = D$ $A = PDP^T$

P is an orthogonal matrix

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