Theory I: Why Deep Structure?

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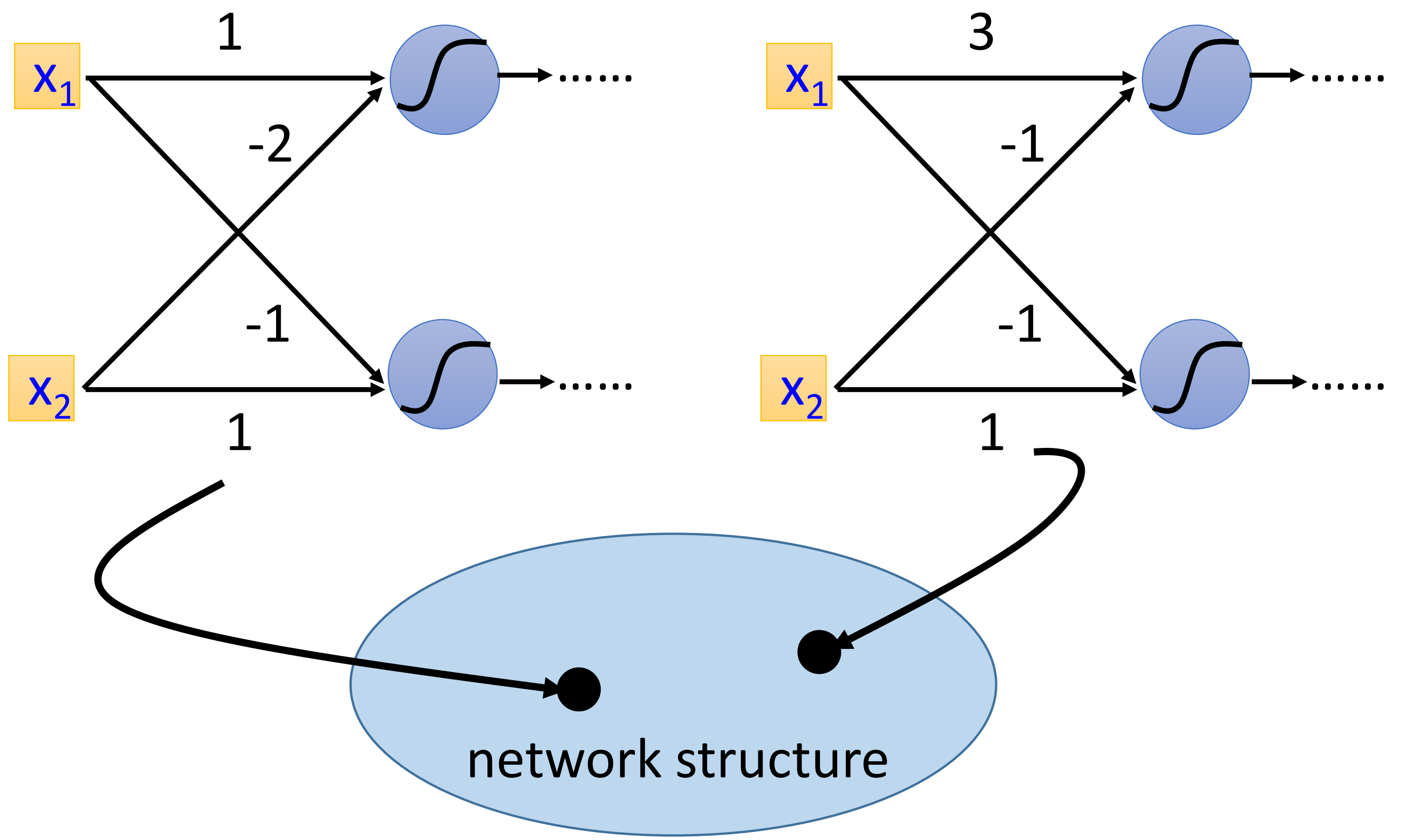
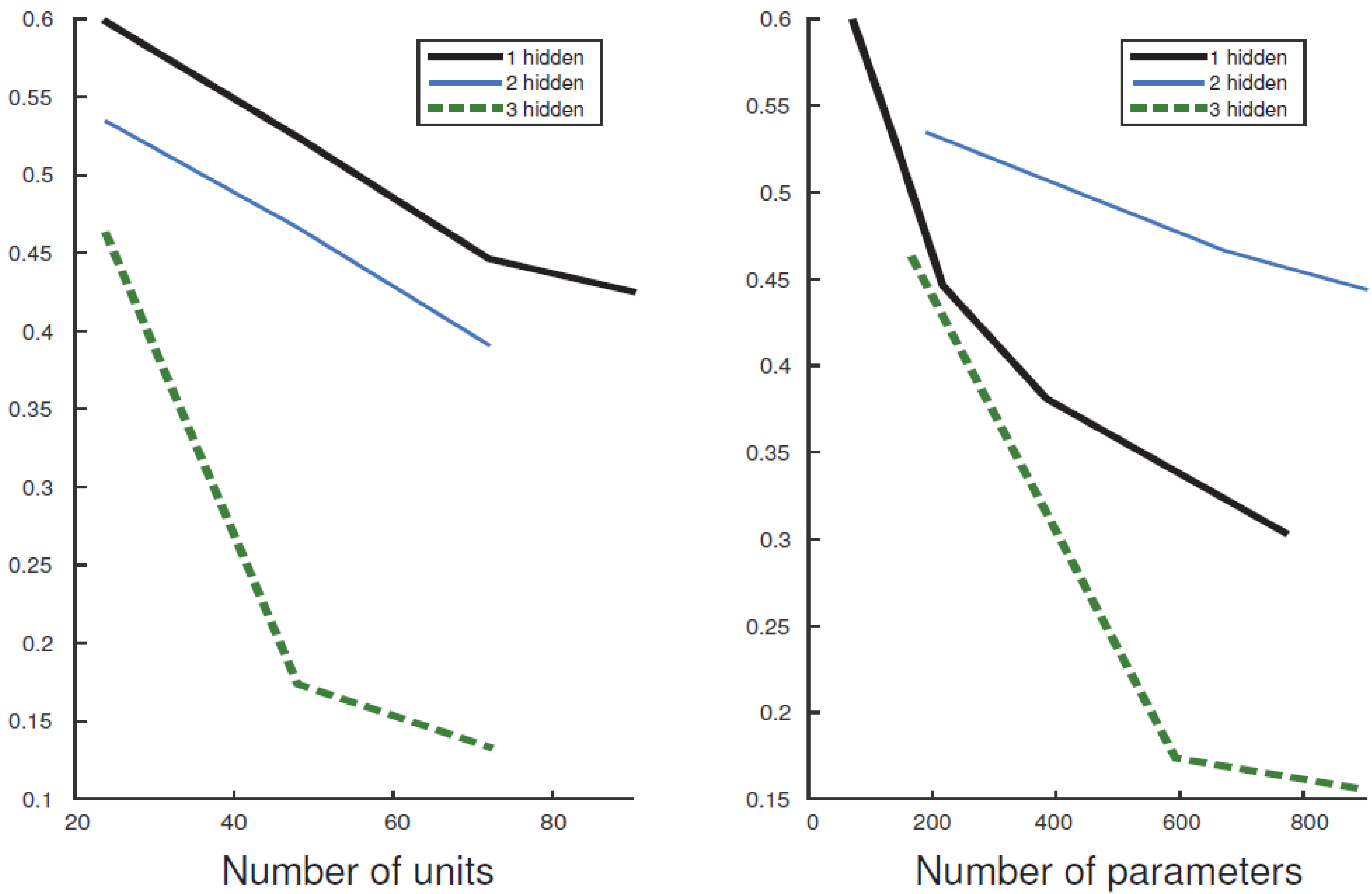
Review



Given structure, each set of parameter is a function.

The network structure defines a function set.

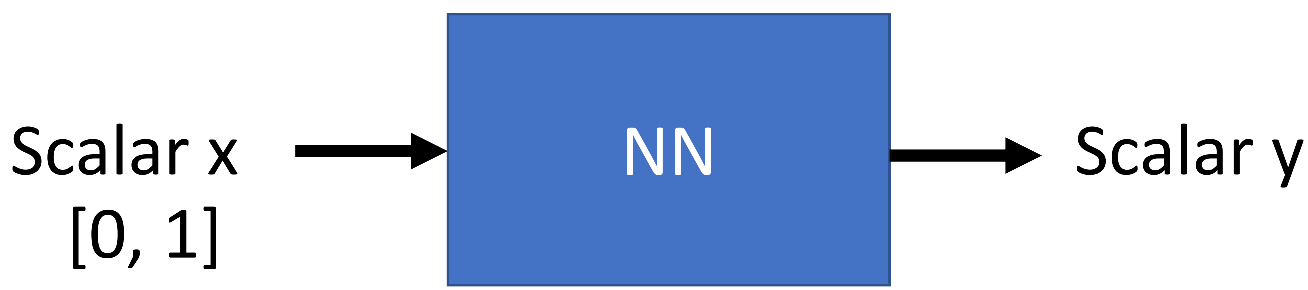
$$f(x) = 2(2\cos^2(x) - 1)^2 - 1$$



Source of image:
https://www.aaai.org/ocs/index.php/AAAI/AAAI17/paper/viewPaper/14849

Outline

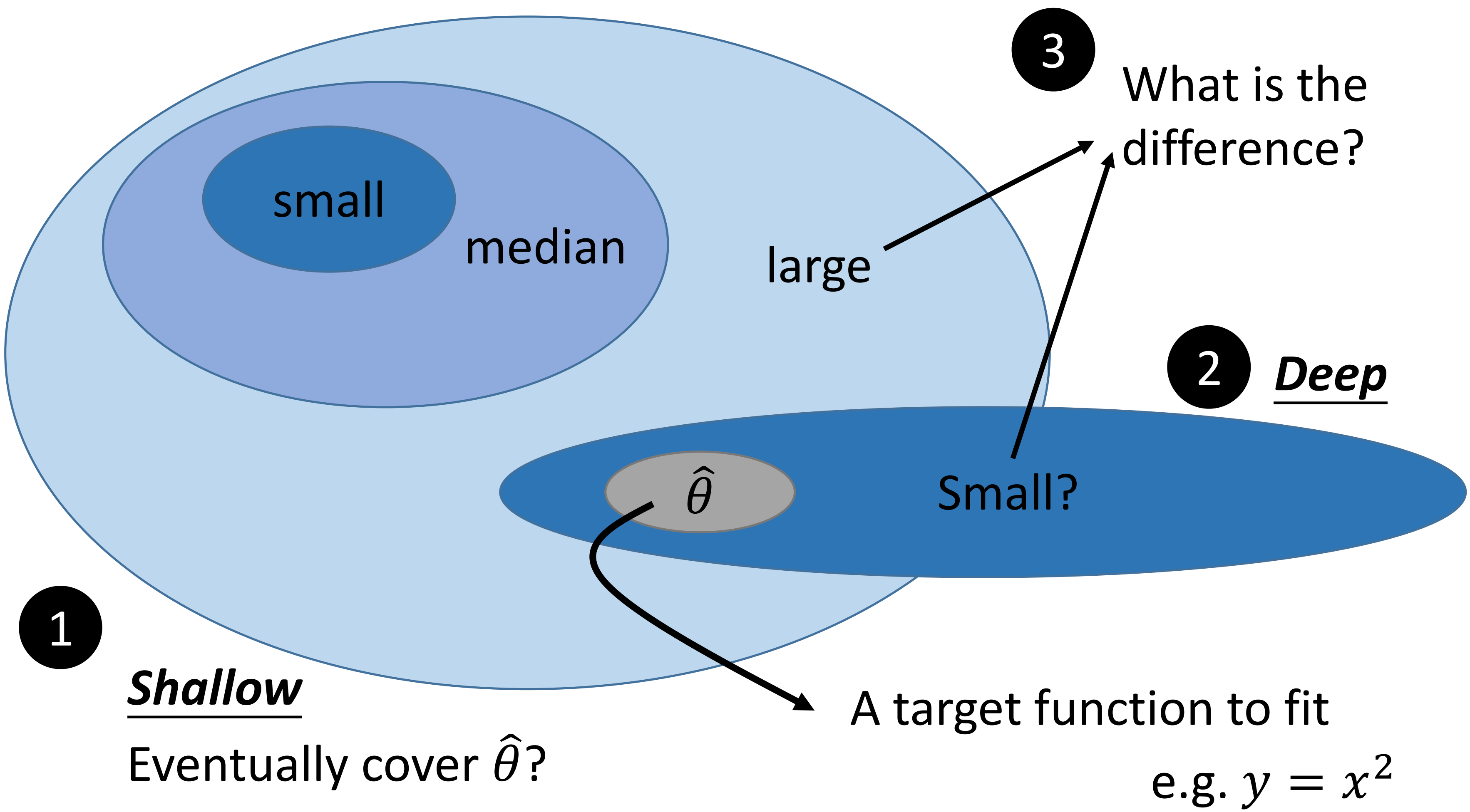
- Q1: Can shallow network fit any function?
- Potential of deep
- Q2: How to use deep to fit functions?
- Q3: Is deep better than shallow?
- Review some related theories



ReLU as activation function

Outline

Notice: We do not discuss ***optimization*** and ***generation*** today.

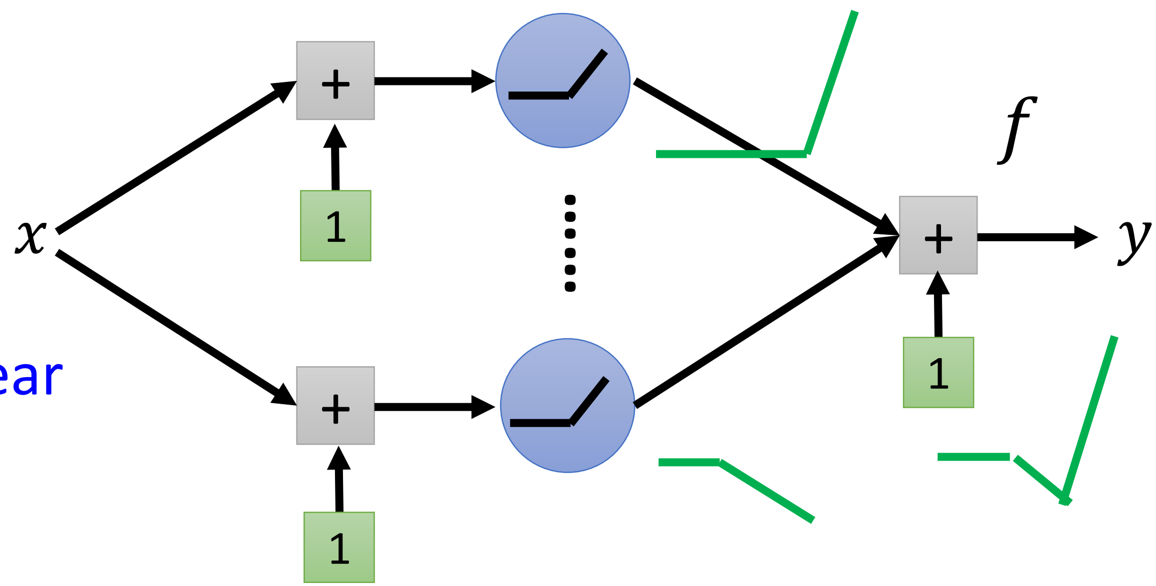


Can shallow network fit any function?

Universality

- Given a ***shallow*** network structure with one hidden layer with ReLU activation and linear output

A piece-wise linear functions



- Given a L-Lipschitz function f^*
  - How many neurons are needed to approximate f^*?

Universality

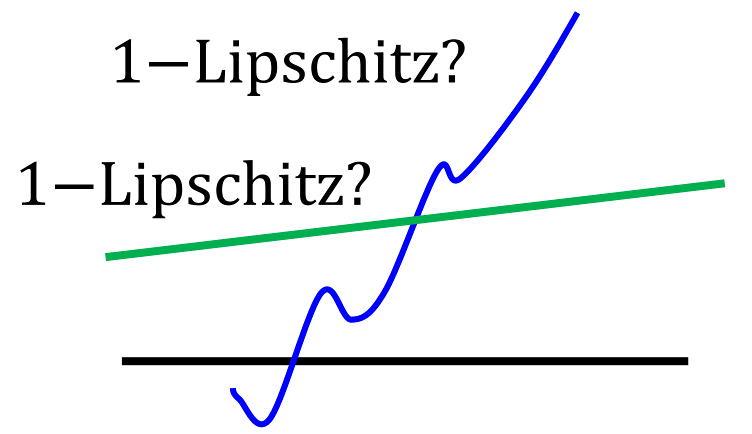
- Given a L-Lipschitz function f^*
  - How many neurons are needed to approximate f^*?

L-Lipschitz Function (smooth)

$$\|f(x_1) - f(x_2)\| \leq L\|x_1 - x_2\|$$

Output change   Input change

L=1 for "$1 - Lipschitz$"



Universality

$$\max_{0 \le x \le 1} |f(x) - f^*(x)| \le \varepsilon$$

$$\sqrt{\int_0^1 |f(x) - f^*(x)|^2 \, dx} \le \varepsilon$$

- Given a L-Lipschitz function f^*
  - How many neurons are needed to approximate f^*?

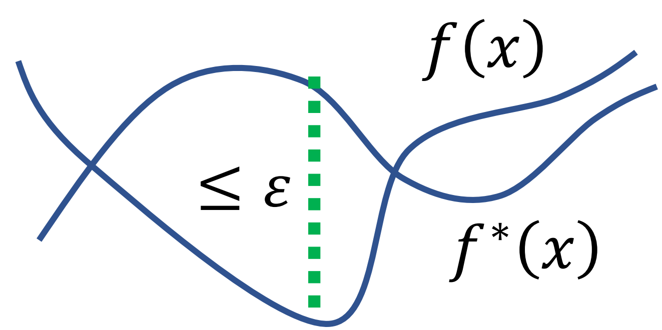
$f \in N(K)$ The function space defined by the network with K neurons.

Given a small number $\varepsilon > 0$

What is the number of K such that

Exist $f \in N(K)$, $\max_{0 \le x \le 1} |f(x) - f^*(x)| \le \varepsilon$

The difference between $f(x)$ and $f^*(x)$ is smaller than ε.



Universality

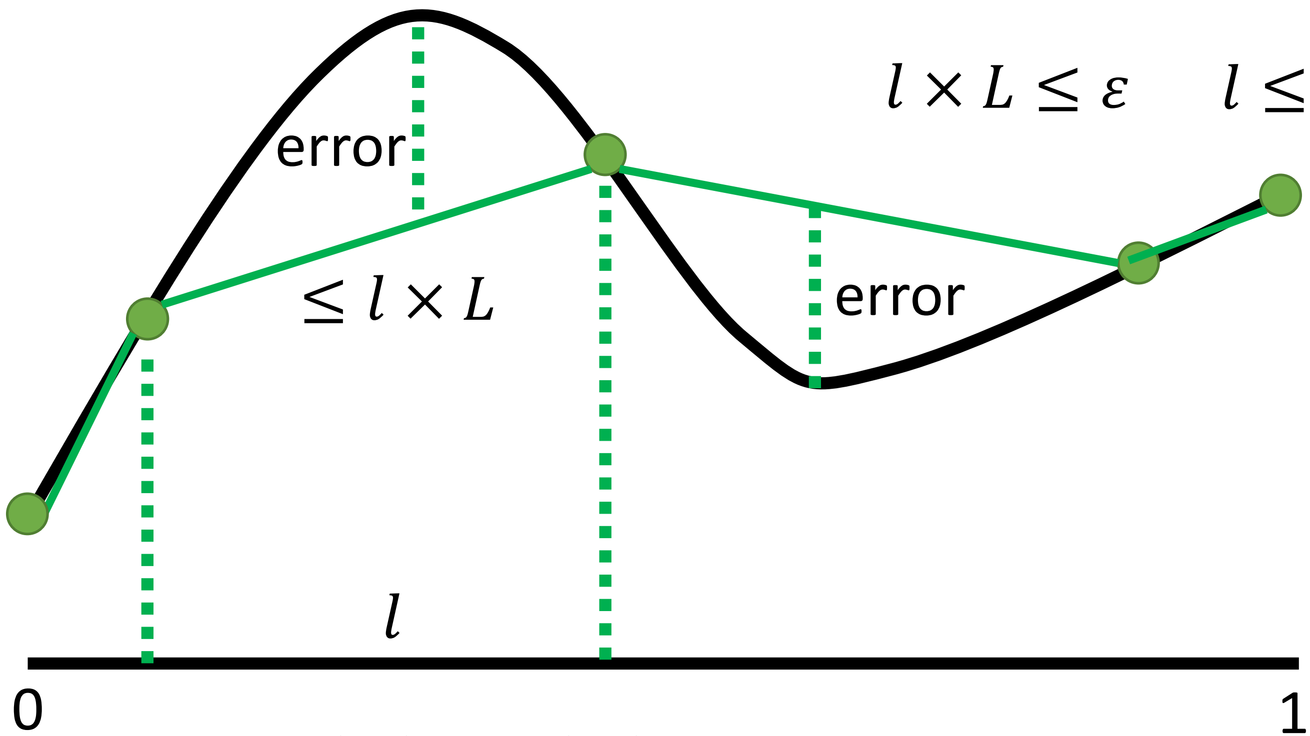
- L-Lipschitz function f^*

All the functions in $N(K)$ are piecewise linear.

Approximate f^* by a piecewise linear function f

How to make the errors $\leq \varepsilon$

$l \times L \leq \varepsilon \qquad l \leq \varepsilon / L$

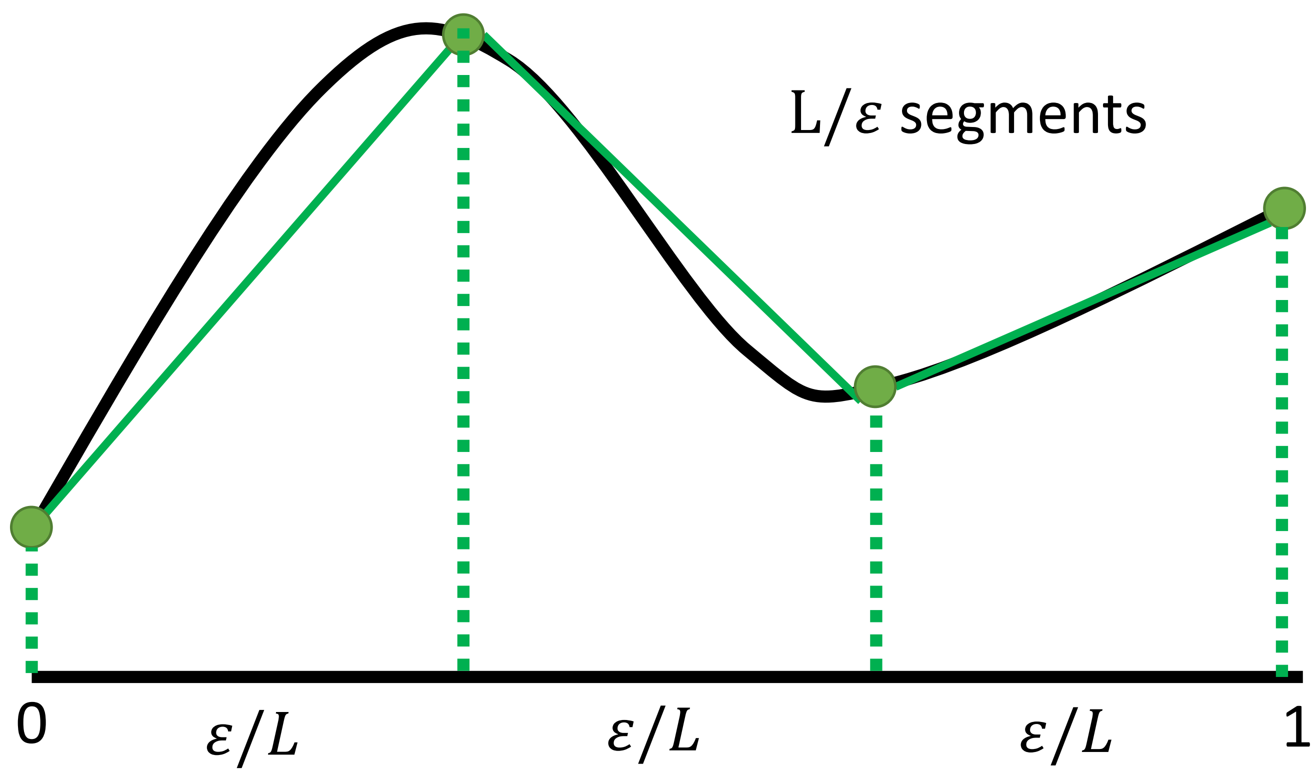


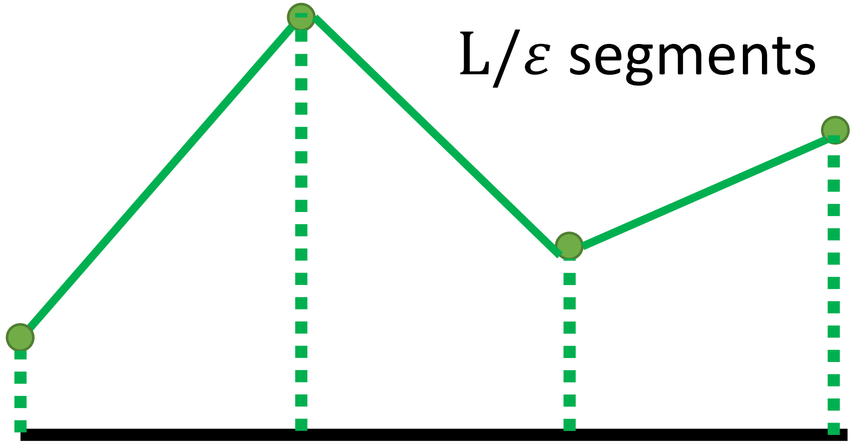
$$\|f(x_1) - f(x_2)\| \leq L\|x_1 - x_2\|$$

Universality

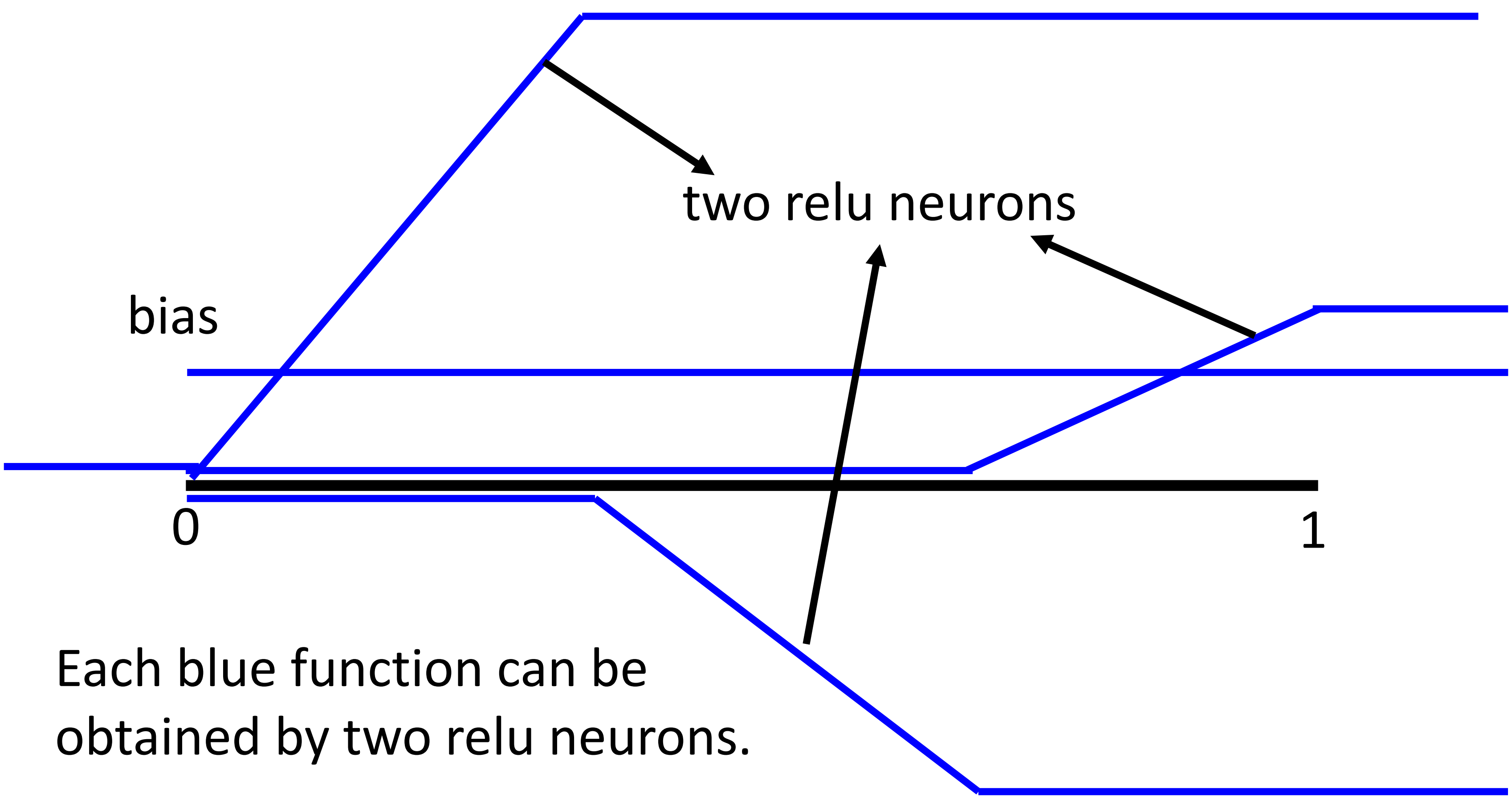
- L-Lipschitz function f^*

How to make a 1 hidden layer relu network have the output like green curve?

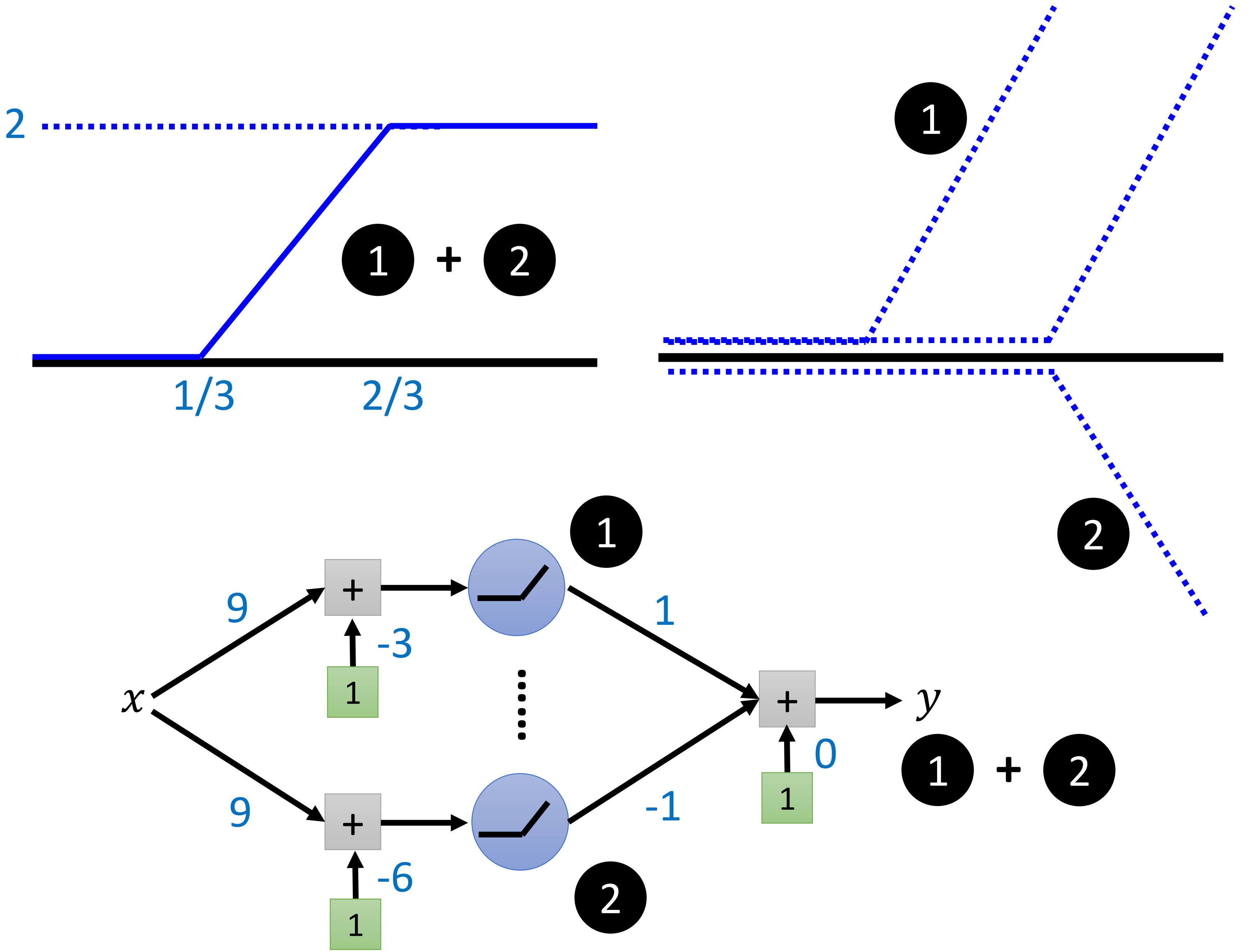




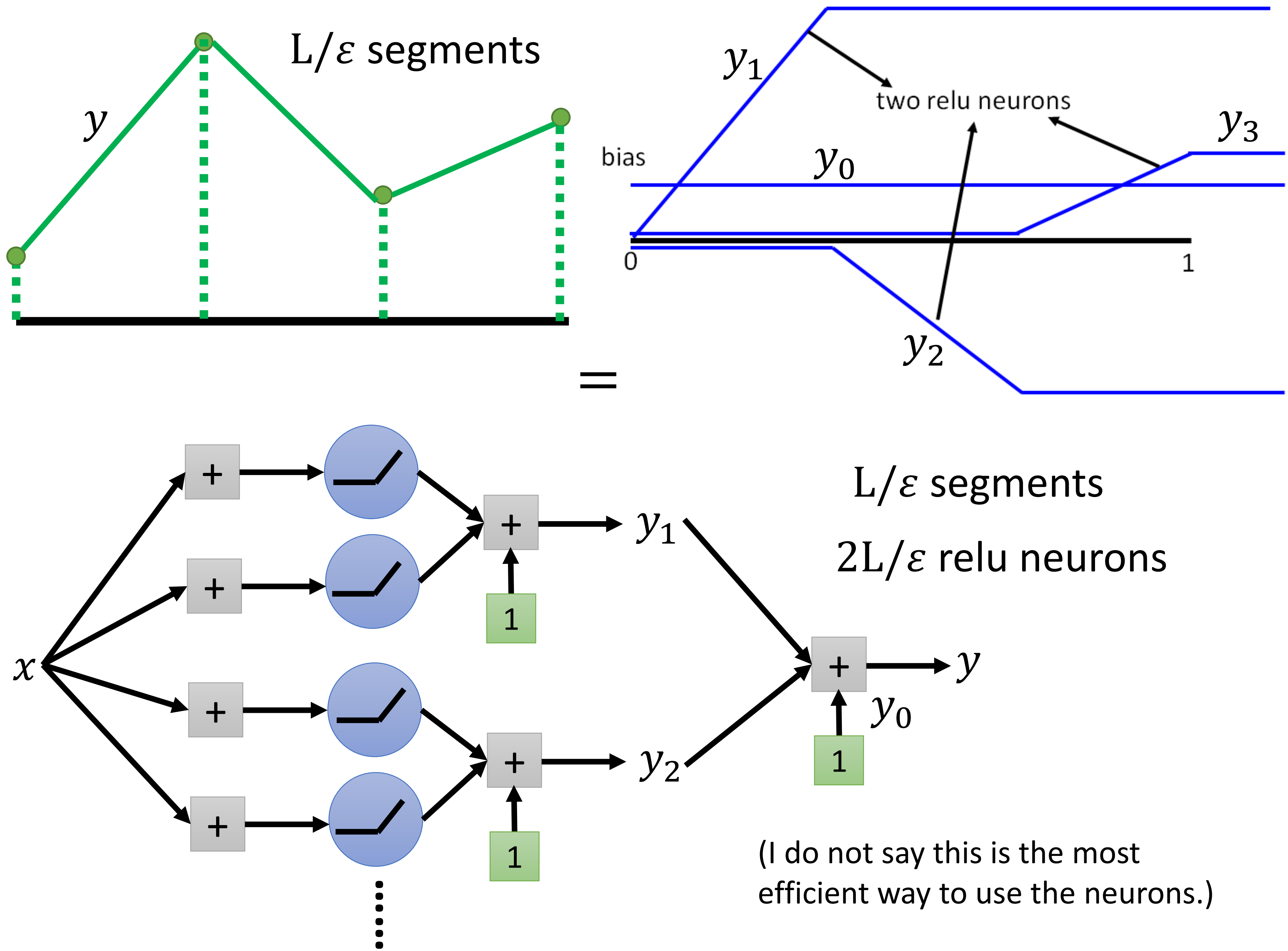
The summation of the blue functions is the green one.



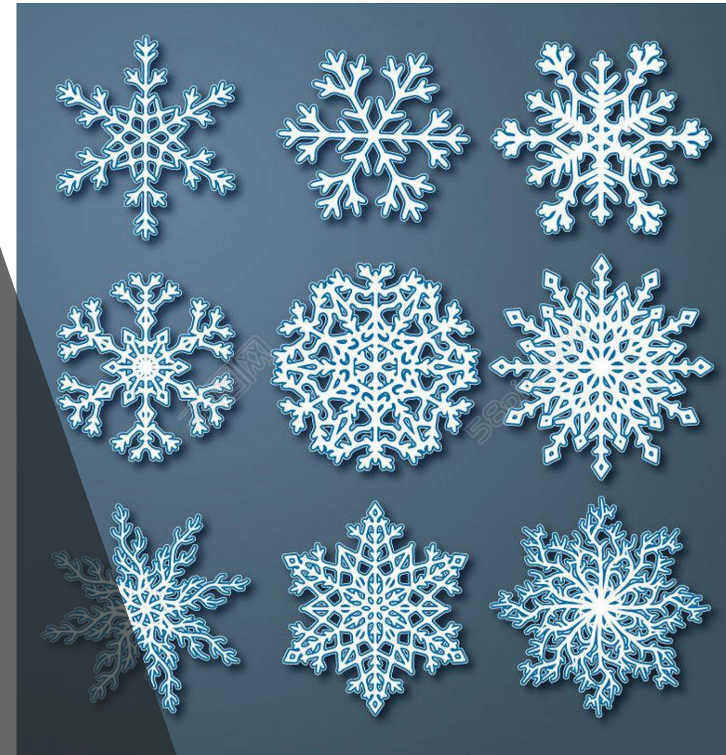
Each blue function can be obtained by two relu neurons.

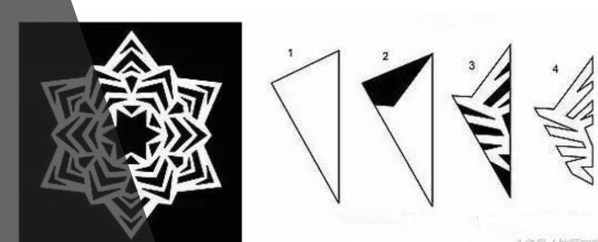


2
1/3
2/3
1 + 2
1
2
x
9
9
+
+
-3
-6
1
1
1
1
-1
+
0
1
y
1 + 2
2

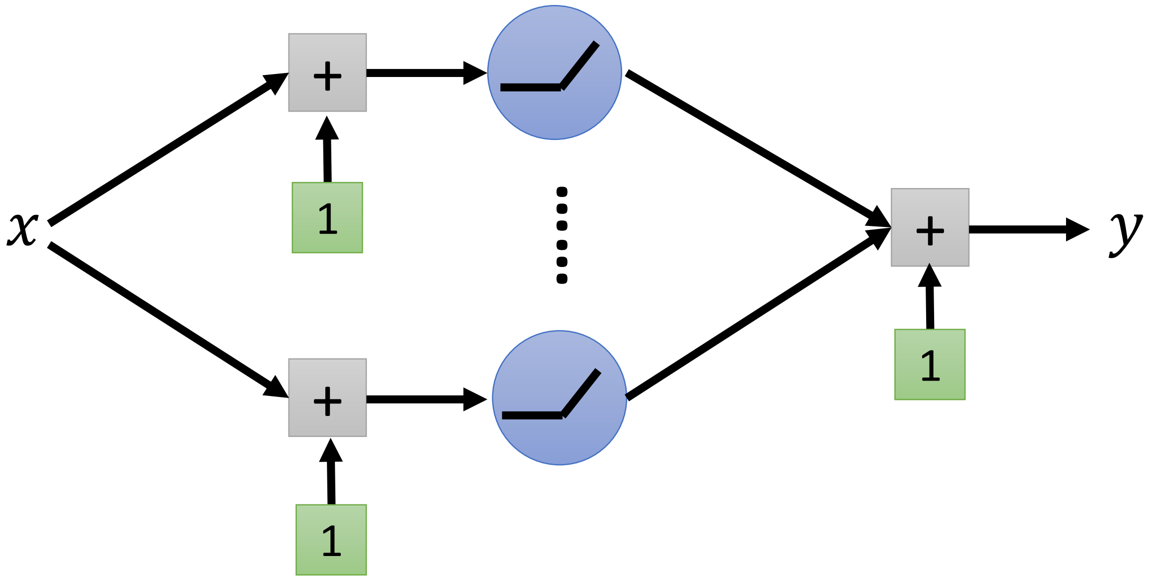

L/ε segments
y
two relu neurons
bias
y₀
y₁
y₂
y₃
0
1
=
x
+
1
y₁
y₂
y
y₀
L/ε segments
2L/ε relu neurons
(I do not say this is the most efficient way to use the neurons.)

Potential of deep





Why we need deep?



Yes, shallow network can represent any function.

However, using deep structure is more effective.

Analogy – Programming

- Solve any problem by two lines (shallow)
  - Input = K
  - Line 1: row no. = MATCH_KEY(K)
  - Line 2: Output the value at row no.

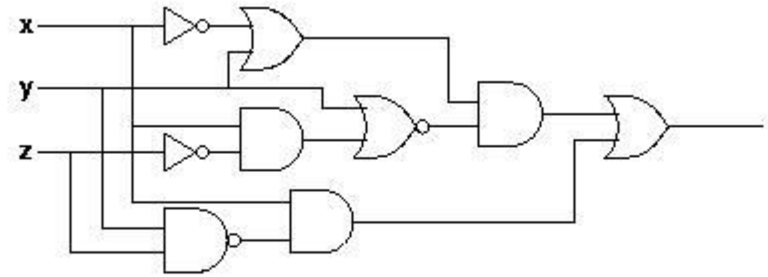
Input (key)	Output (value)
A	A’
B	B’
C	C’
D	D’
……	……

- Considering SVM with kernel

$$y = \sum_n \alpha_n K(x^n, x)$$

- Using multiple steps to solve problems is more efficient (deep)

Analogy



Logic circuits

- Logic circuits consists of **gates**
- **A two layers of logic gates** can represent **any Boolean function.**
- Using multiple layers of logic gates to build some functions are much simpler

→ less gates needed

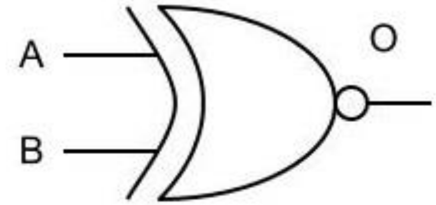
Neural network

- Neural network consists of **neurons**
- **A hidden layer network** can represent **any continuous function.**
- Using multiple layers of neurons to represent some functions are much simpler

→ less neurons

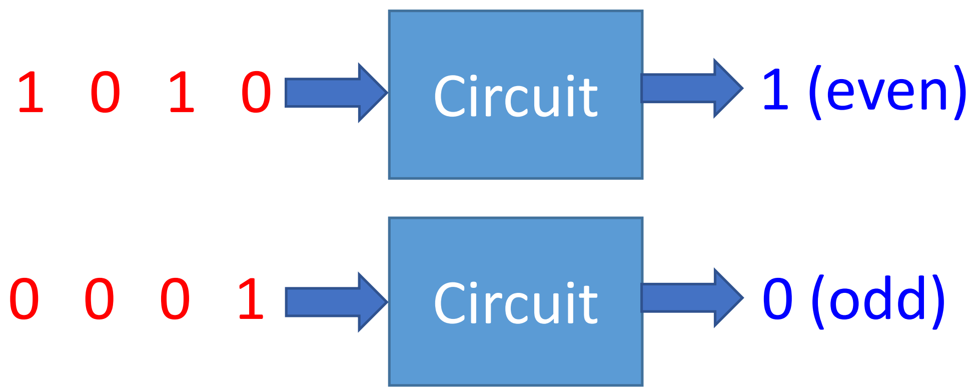
This page is for EE background.

Analogy



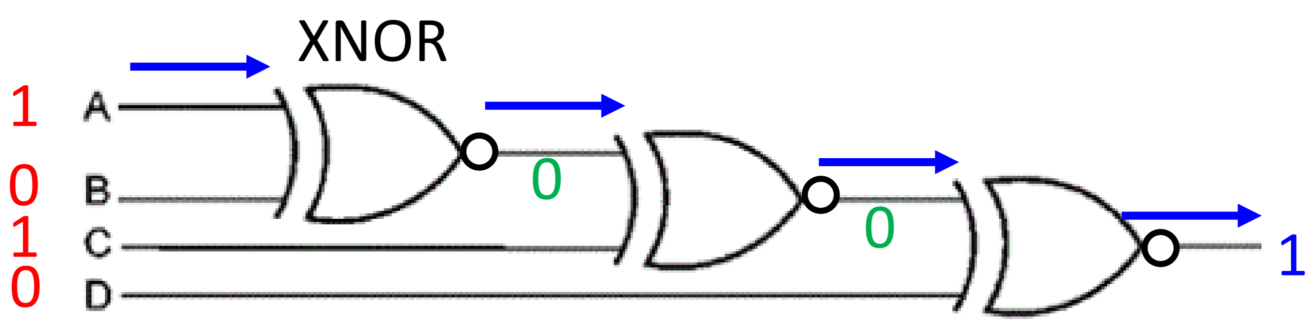
A	B	O
0	0	1
0	1	0
1	0	0
1	1	1

- E.g. ***parity check***



For input sequence with d bits,

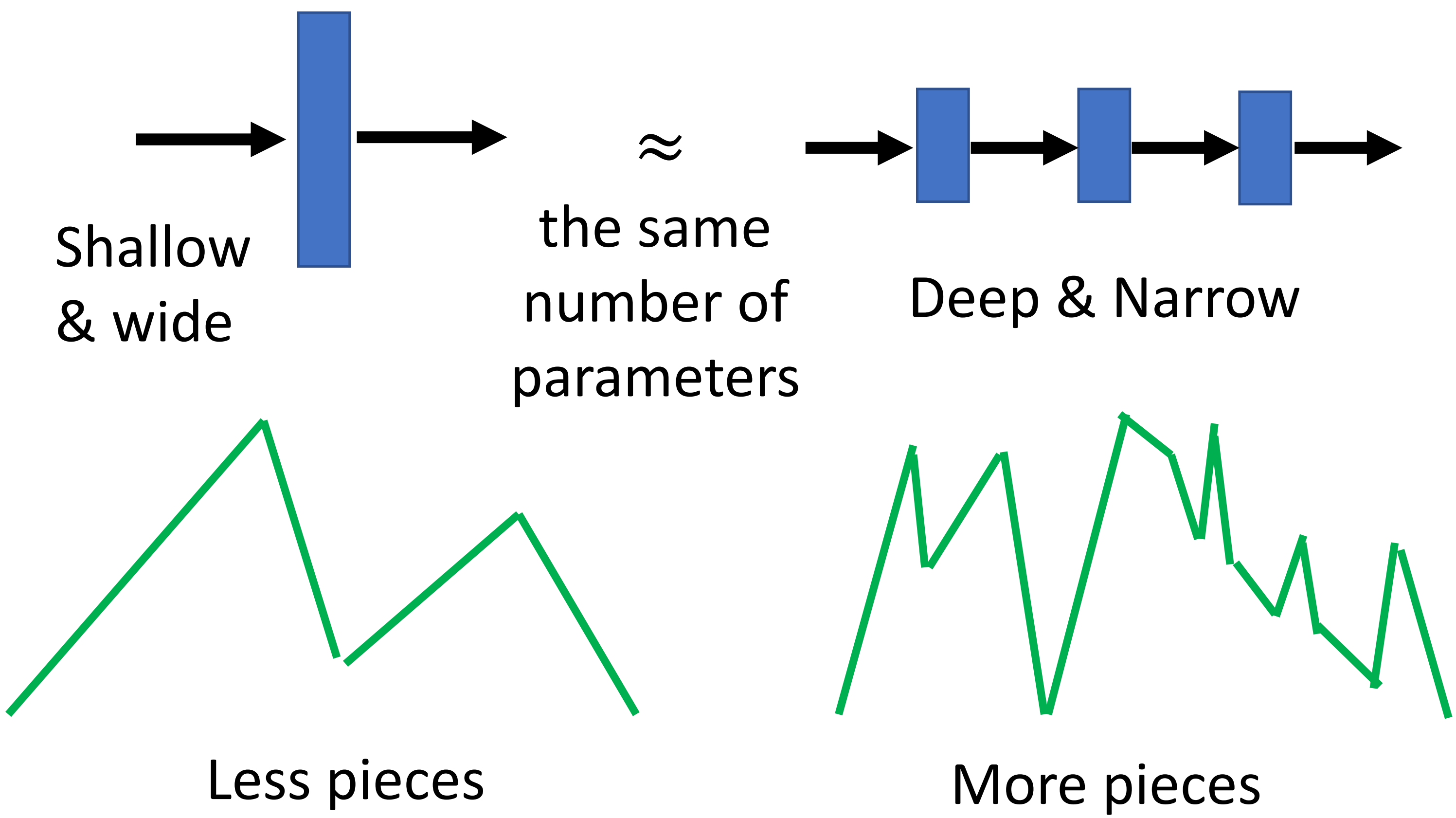
Two-layer circuit need $O(2^d)$ gates.



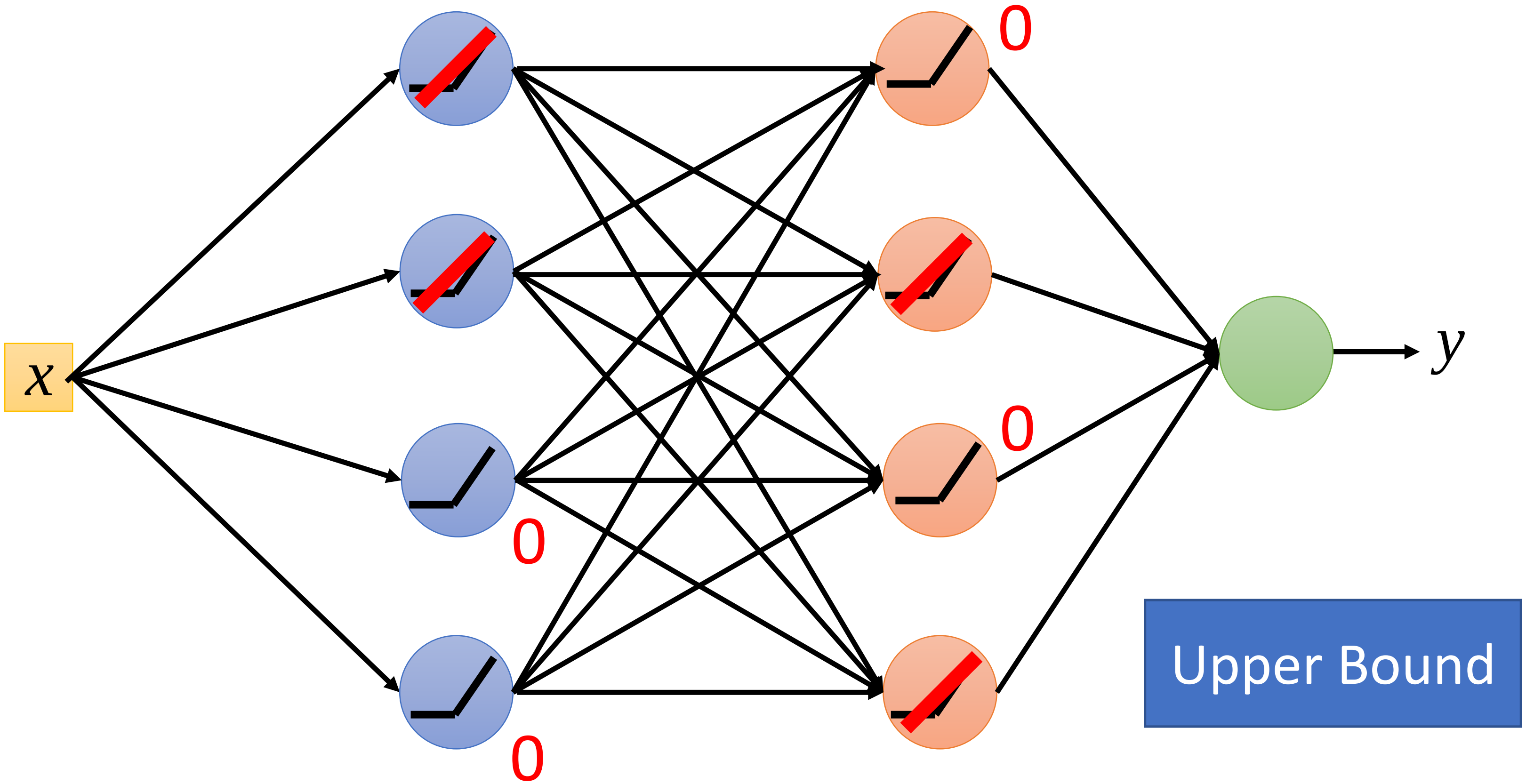
With multiple layers, we need only O(d) gates.

Why we need deep?

- ReLU networks can represent piecewise linear functions



Upper Bound of Linear Pieces

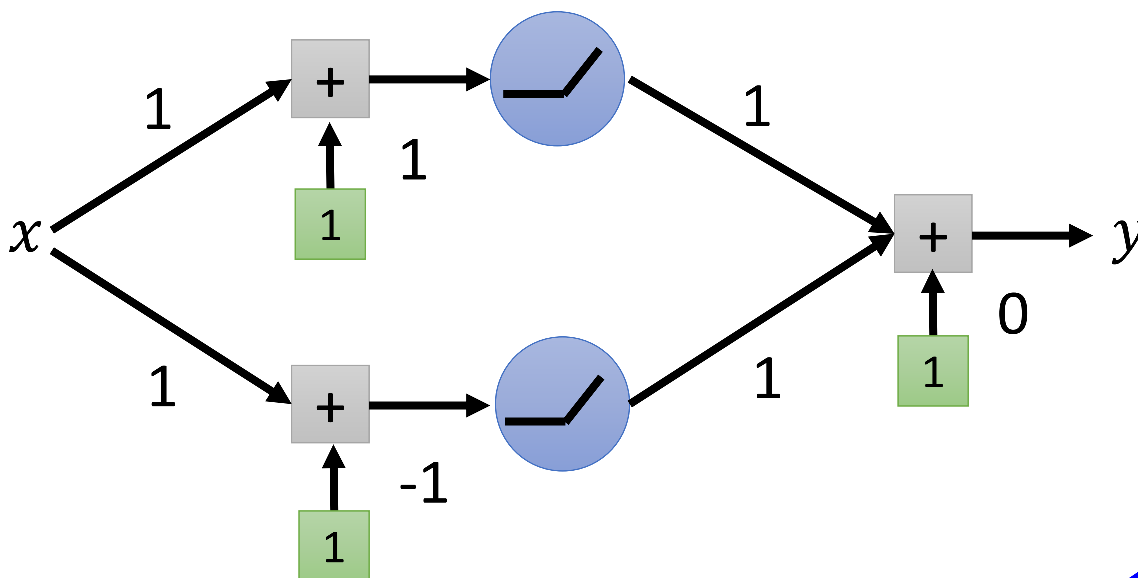


Each “activation pattern” defines a linear function.

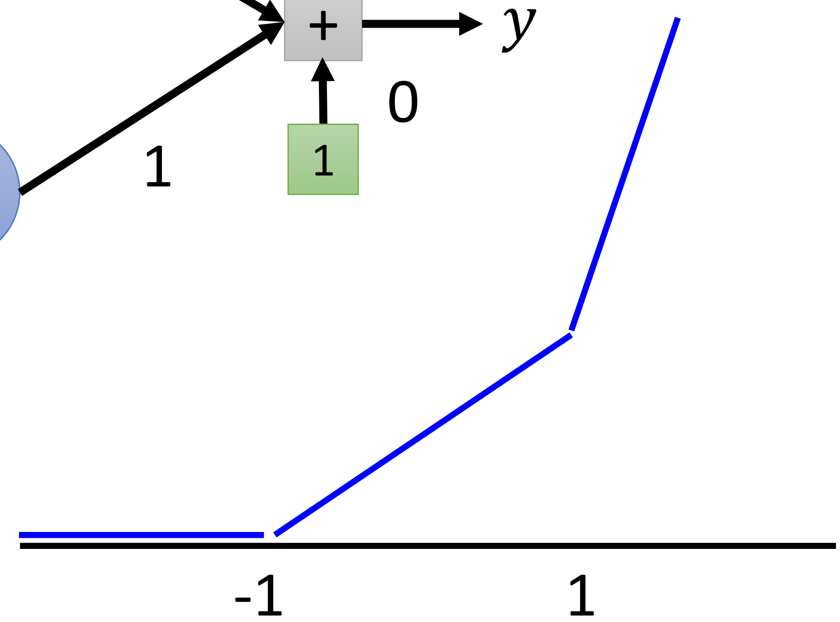
N neurons ➡ 2^N “activation patterns” ➡ 2^N “linear pieces”

Upper Bound of Linear Pieces

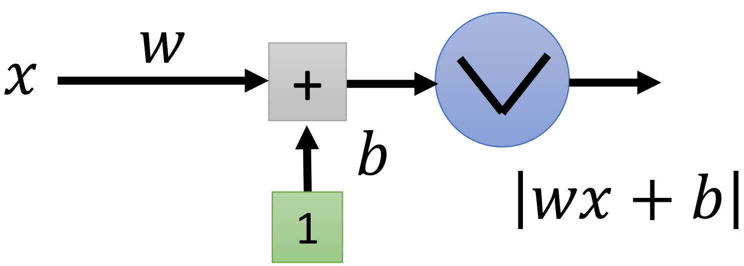
- Not all the “activation patterns” available

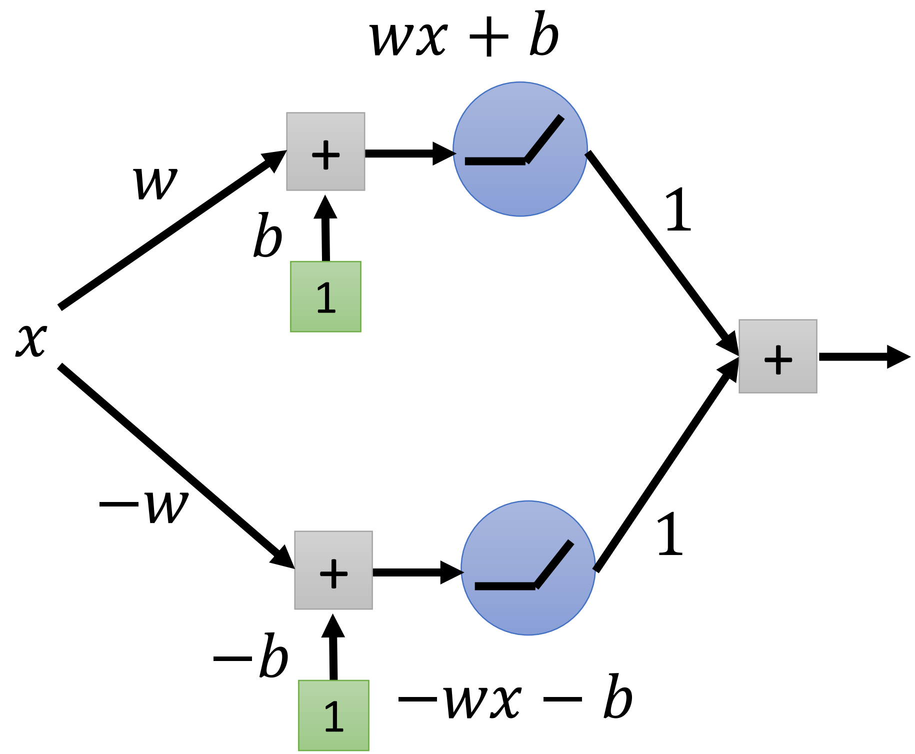


In shallow network, each neuron only provides one linear piece.

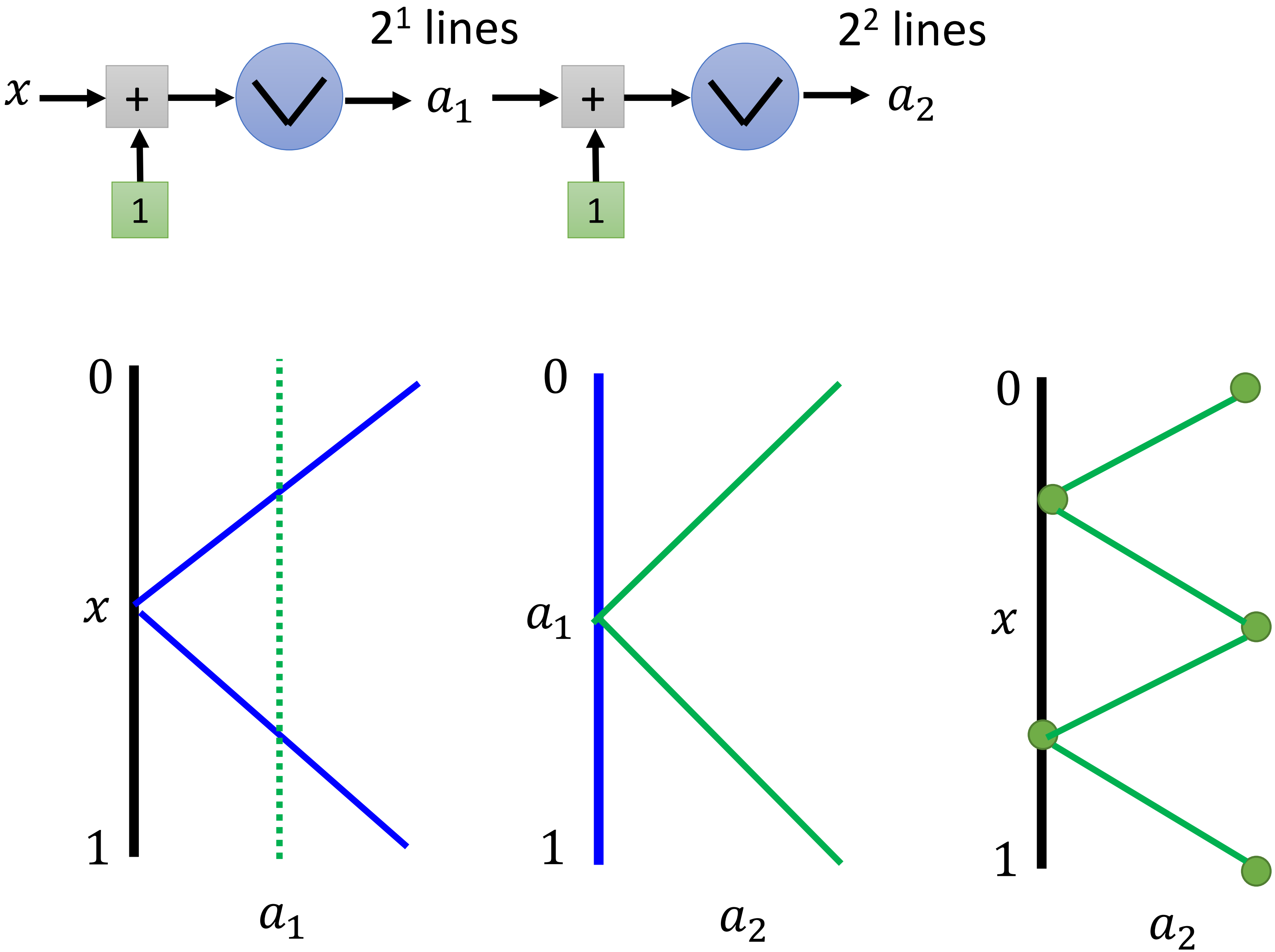


Abs Activation Function



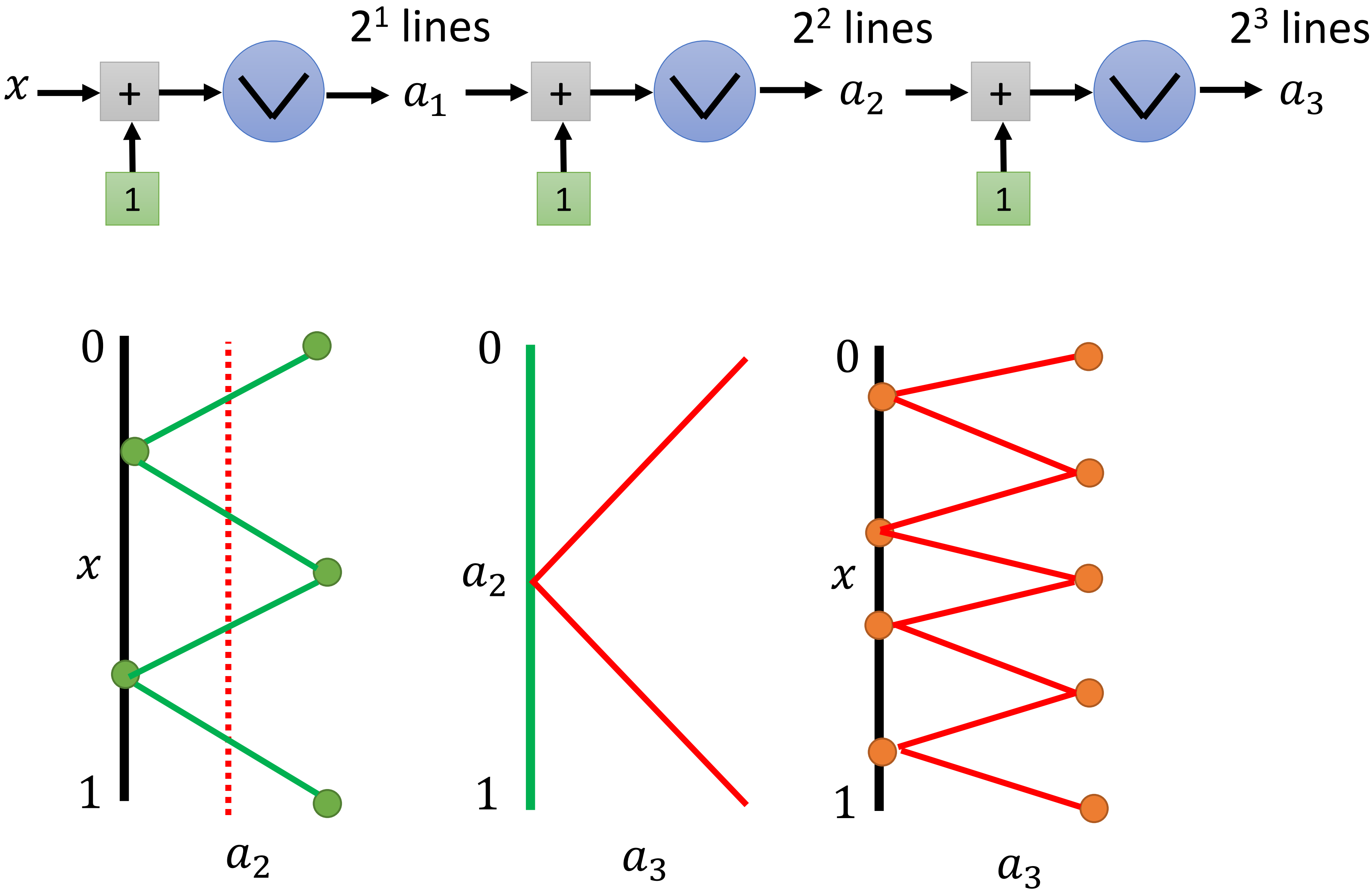


Use two relu to implement
an abs activation function

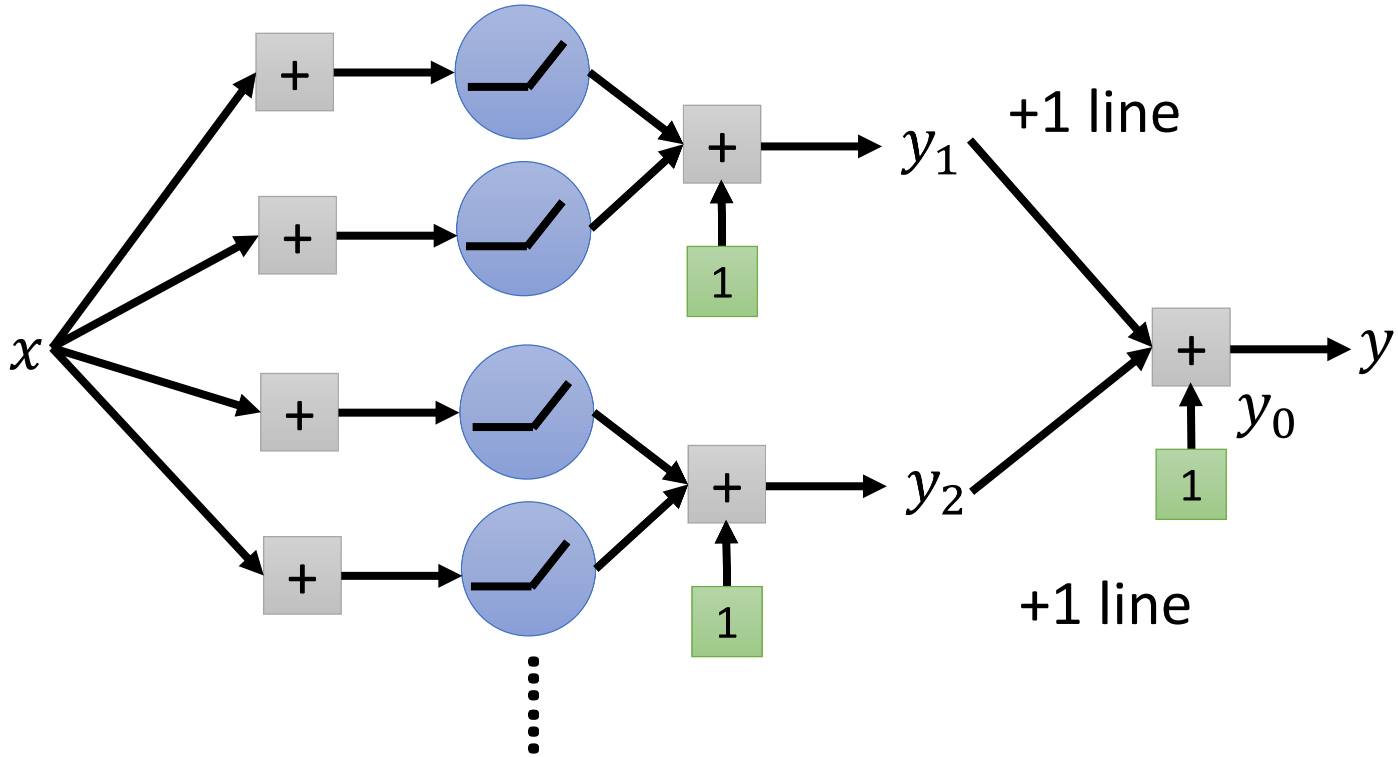


2^1 lines
2^2 lines
x
+
1
a_1
+
1
a_2
0
x
1
a_1
0
a_1
1
a_2
0
x
1
a_2

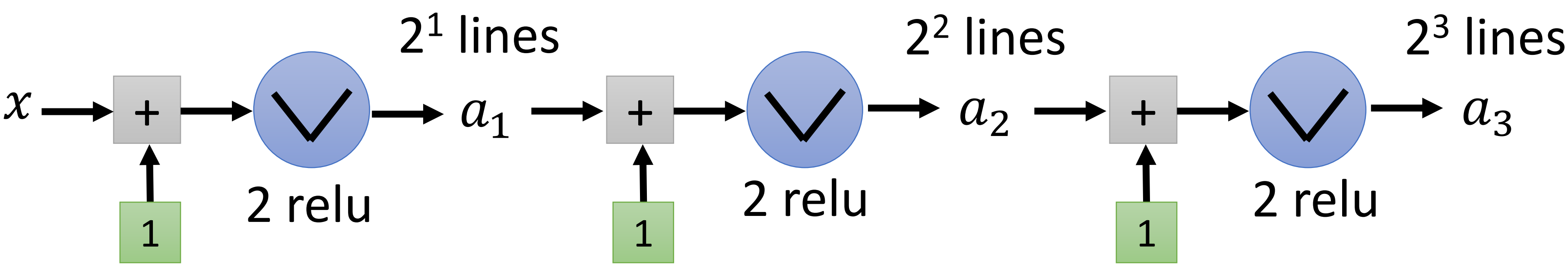
Each node added ⇨ The regions are twice.



Shallow



Deep



Lower Bound of Linear Pieces

If K is width, H is depth

We can have at least K^H pieces

Depth has much larger influence than depth.

Razvan Pascanu, Guido Montufar, Yoshua Bengio, "On the number of response regions of deep feed forward networks with piece-wise linear activations", ICLR, 2014

Guido F. Montufar, Razvan Pascanu, Kyunghyun Cho, Yoshua Bengio, "On the Number of Linear Regions of Deep Neural Networks", NIPS, 2014

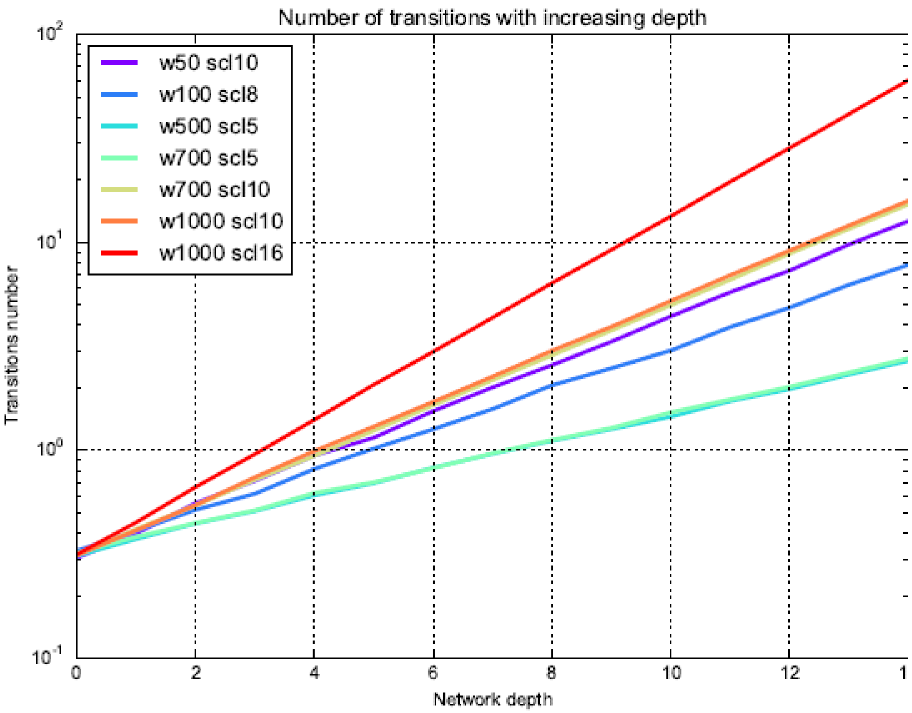
Raman Arora, Amitabh Basu, Poorya Mianjy, Anirbit Mukherjee, "Understanding Deep Neural Networks with Rectified Linear Units", ICLR 2018

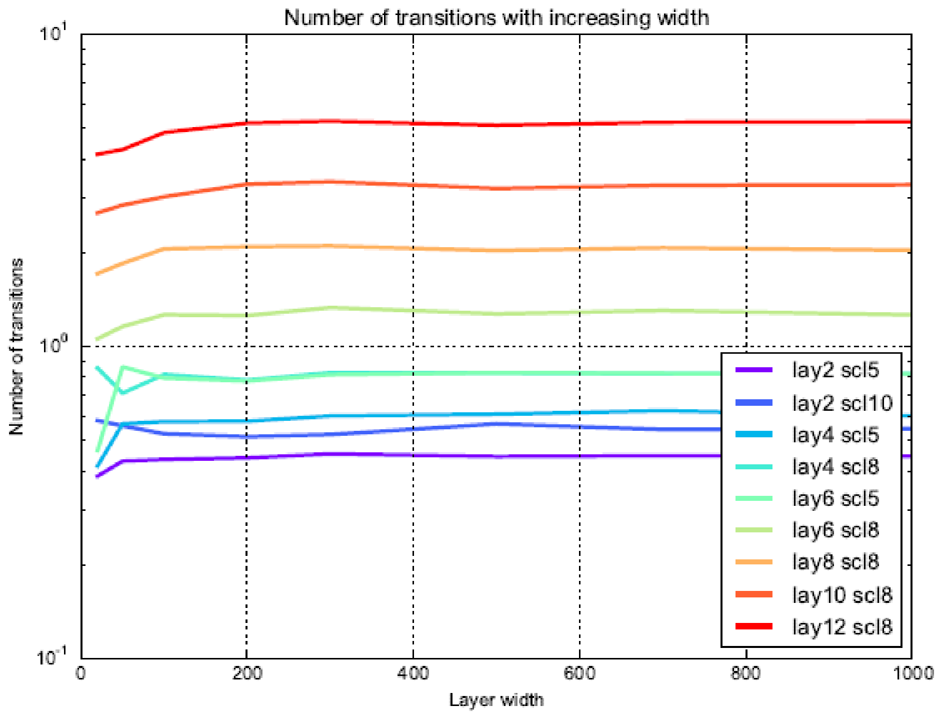
Thiago Serra, Christian Tjandraatmadja, Srikumar Ramalingam, "Bounding and Counting Linear Regions of Deep Neural Networks", arXiv, 2017

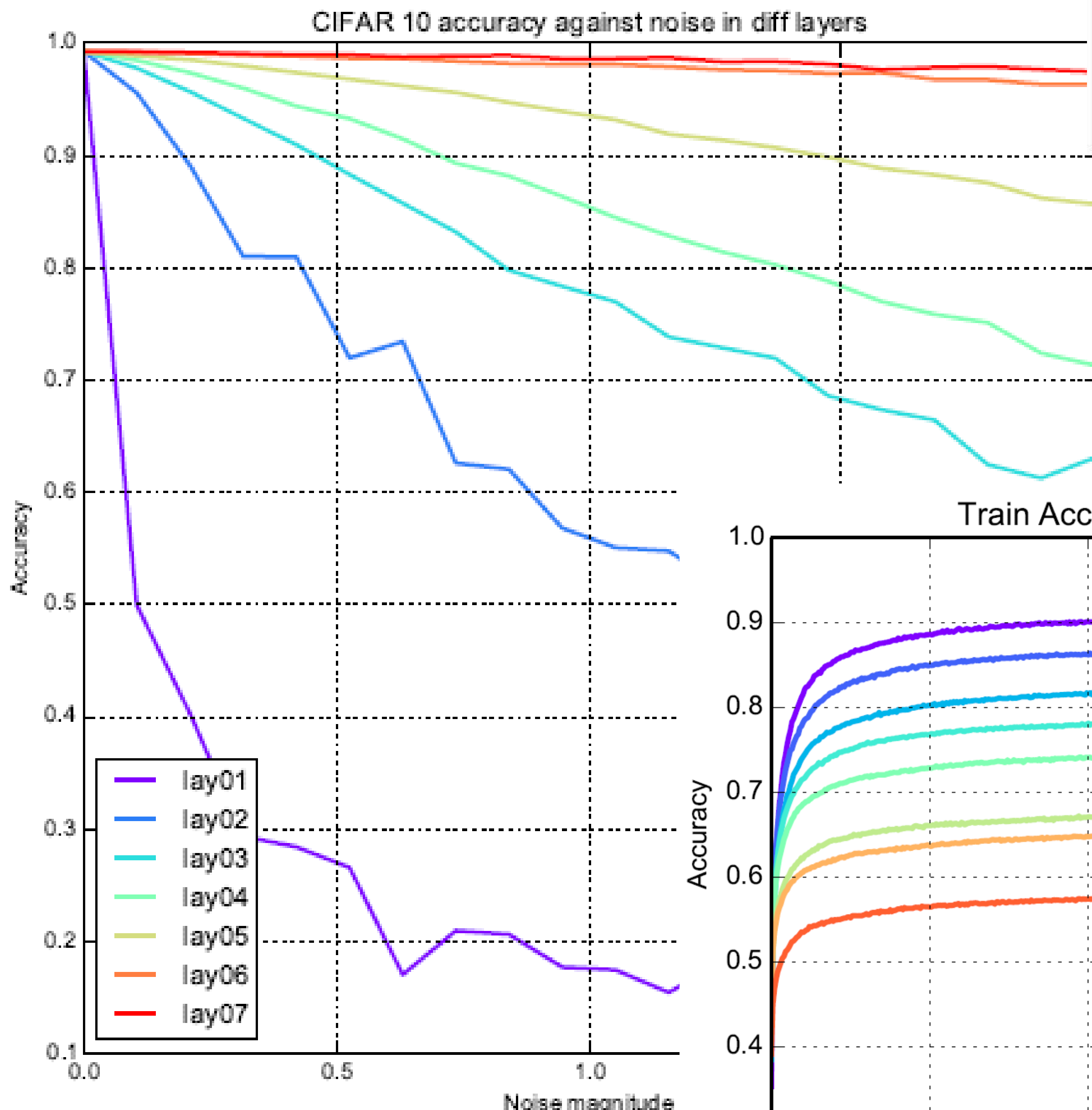
Maithra Raghu, Ben Poole, Jon Kleinberg, Surya Ganguli, Jascha Sohl-Dickstein, On the Expressive Power of Deep Neural Networks, ICML, 2017

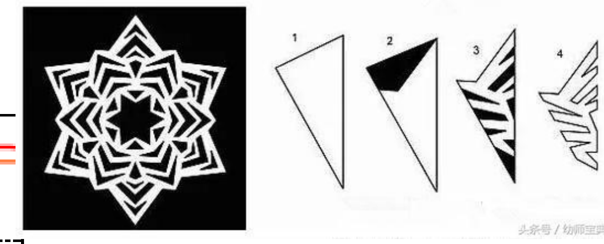
Experimental Results

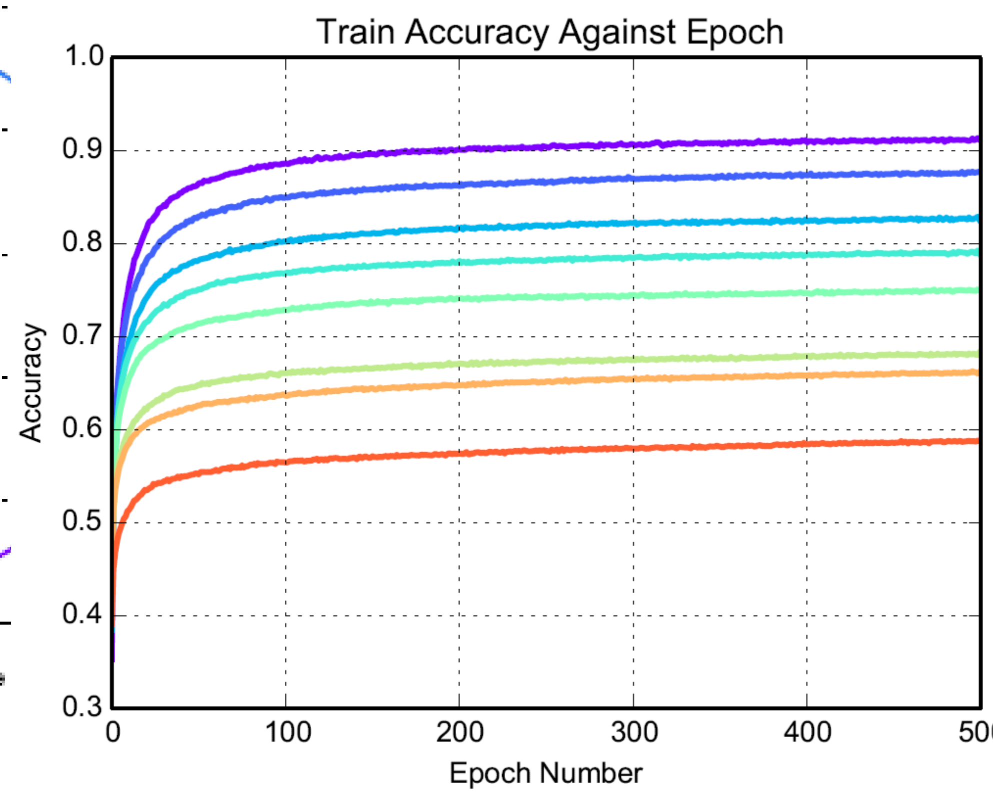
(MNIST)





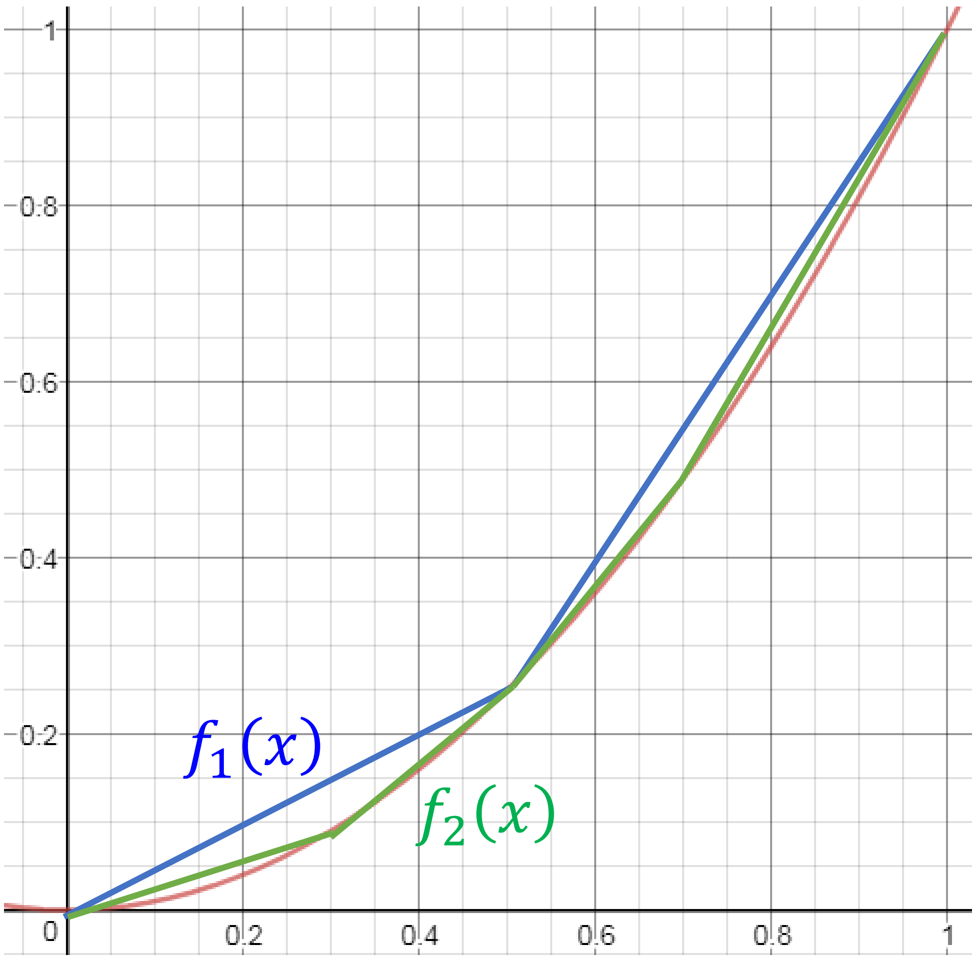

CIFAR 10 accuracy against noise in diff layers
Accuracy
Noise magnitude
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.5
1.0
lay01
lay02
lay03
lay04
lay05
lay06
lay07




Train Accuracy Against Epoch
Accuracy
Epoch Number
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0
100
200
300
400
500

How much is deep better than shallow?

$f(x) = x^2$



Fit the function by equally spaced linear pieces

$f_m(x)$: a function with 2^m pieces

$$\max_{0 \le x \le 1} |f(x) - f_m(x)| \le \varepsilon$$

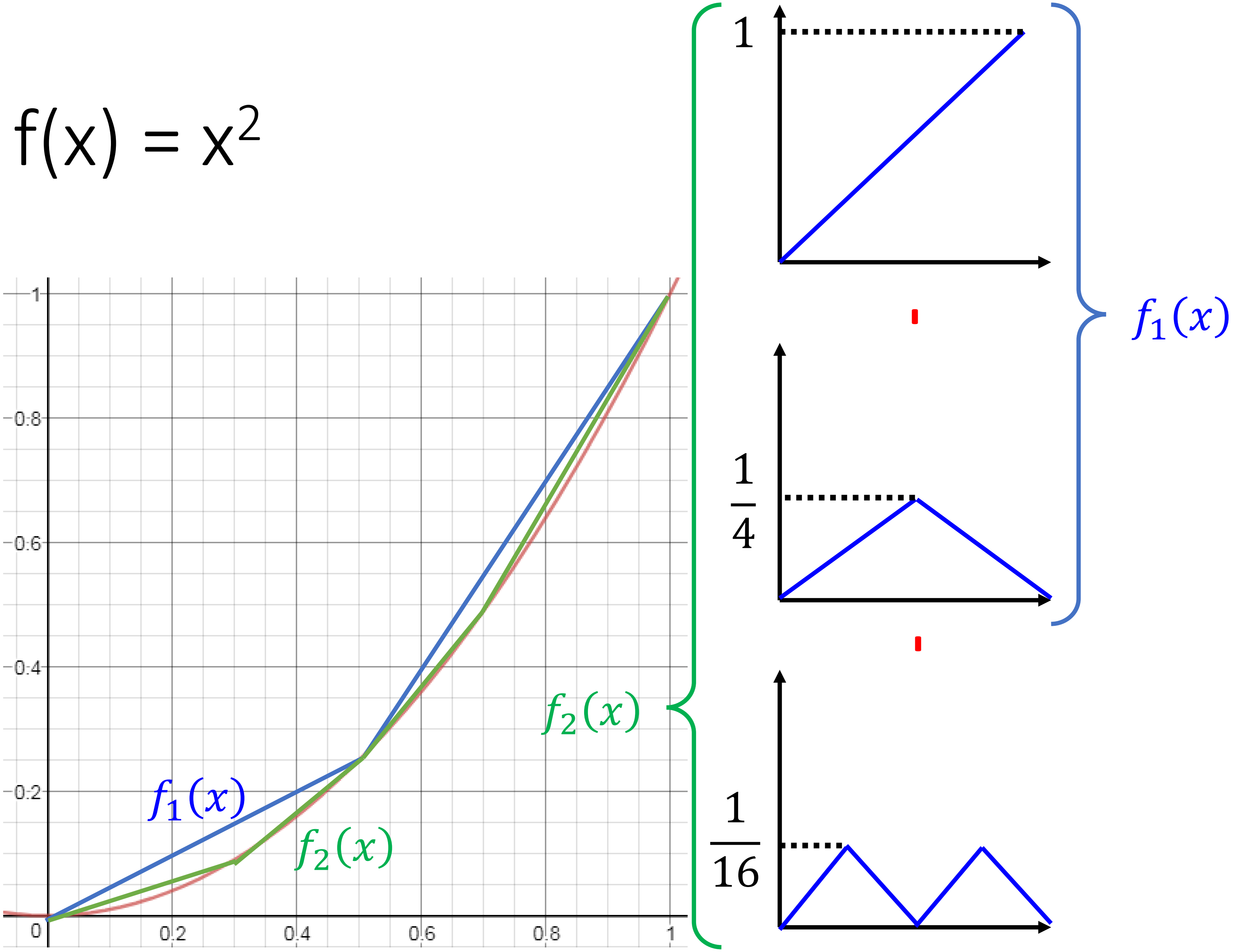
What is the minimum m?

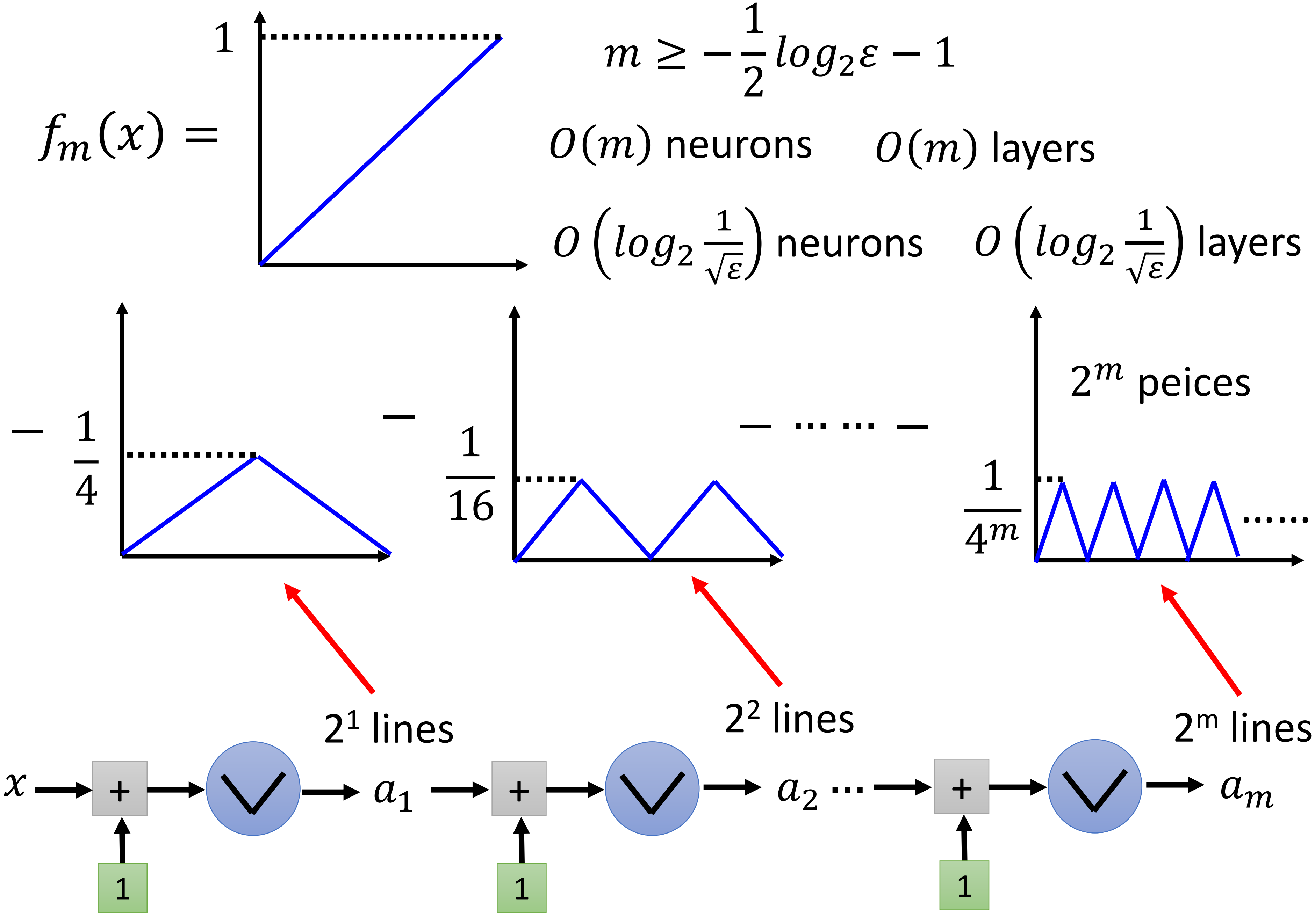
$$m \ge -\frac{1}{2} log_2 \varepsilon - 1$$

$$2^m \ge \frac{1}{2}\frac{1}{\sqrt{\varepsilon}} \text{ pieces}$$

Shallow: $O\left(\frac{1}{\sqrt{\varepsilon}}\right)$ neurons

$$f(x) = x^2$$



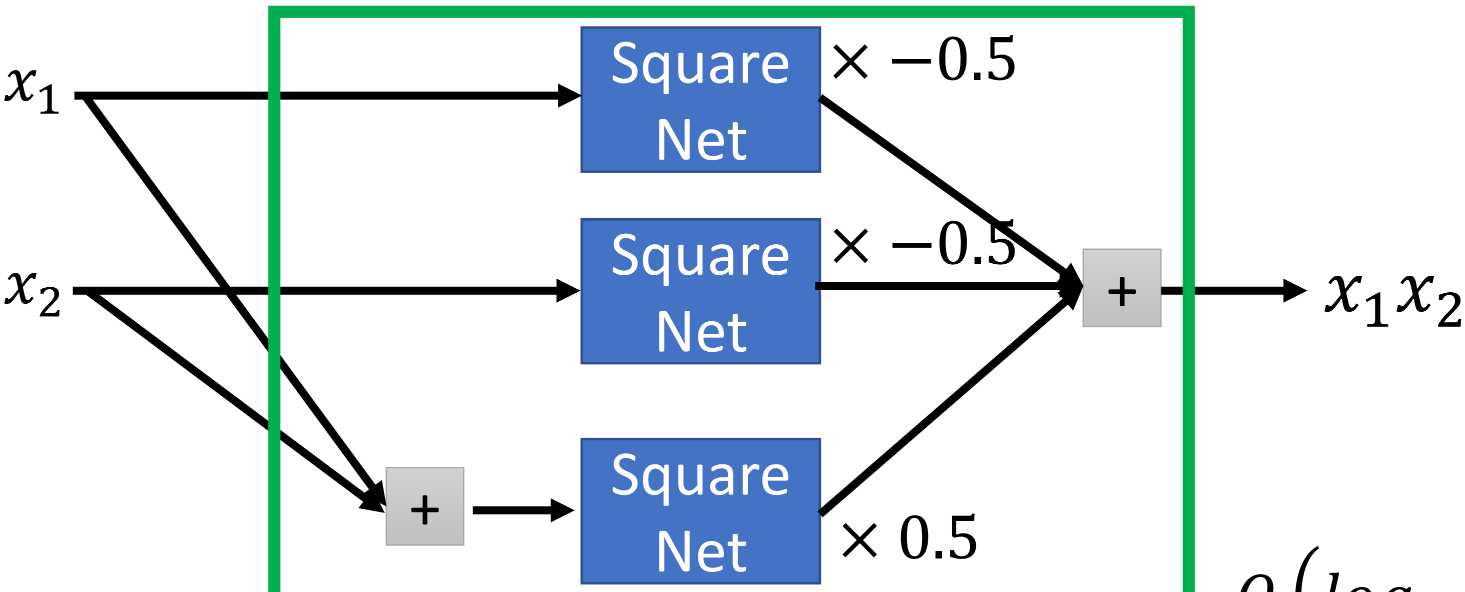


$m \geq -\frac{1}{2}log_2\varepsilon - 1$
$f_m(x) =$
1
$O(m)$ neurons
$O(m)$ layers
$O\left(log_2\frac{1}{\sqrt{\varepsilon}}\right)$ neurons
$O\left(log_2\frac{1}{\sqrt{\varepsilon}}\right)$ layers
$-$
$\frac{1}{4}$
$-$
$\frac{1}{16}$
$- \cdots\cdots -$
2^m peices
$\frac{1}{4^m}$
2^1 lines
2^2 lines
2^m lines
x
+
1
a_1
+
1
$a_2 \cdots$
+
1
a_m

Why care about $y = x^2$?

$O\left(log_2 \frac{1}{\sqrt{\varepsilon}}\right)$ neurons

$$y = x_1 x_2$$
$$= \frac{1}{2}\left((x_1 + x_2)^2 - x_1^2 - x_2^2\right)$$



Multiply Net

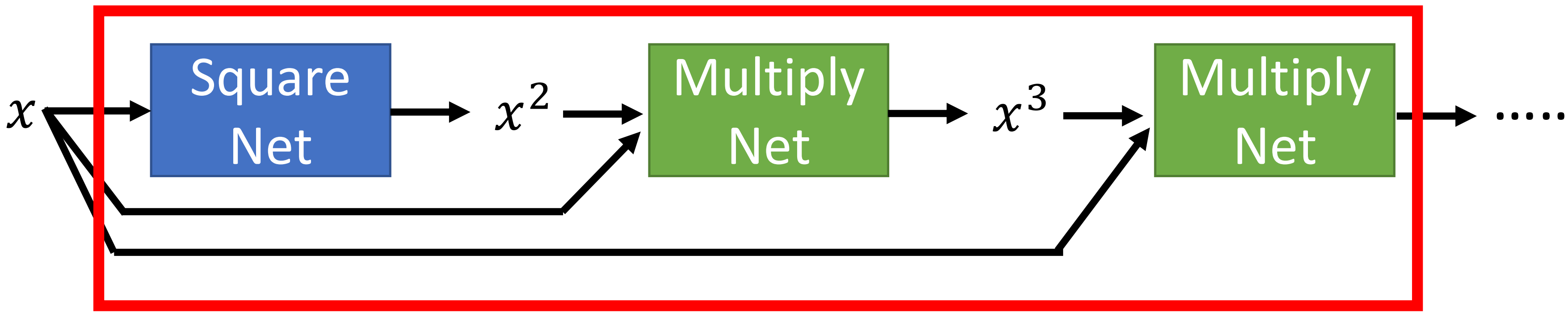
$O\left(log_2 \frac{1}{\sqrt{\varepsilon}}\right)$ neurons

Polynomial

$y = x^n$

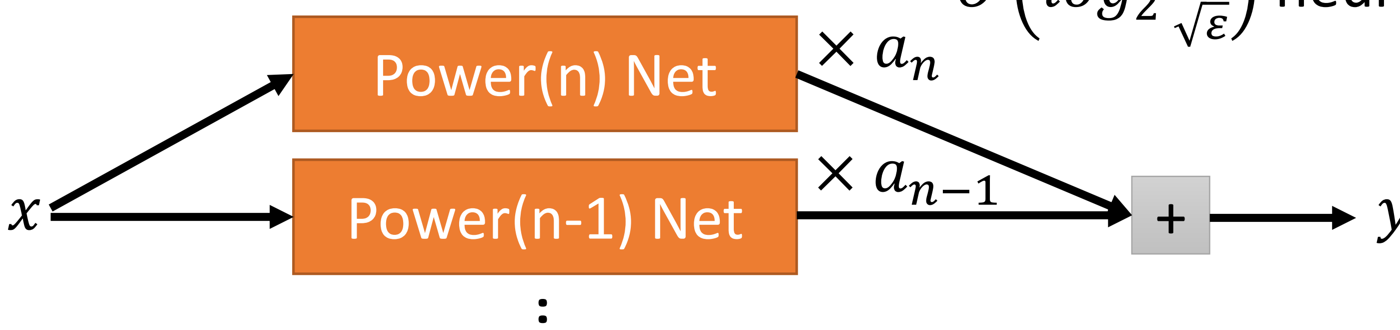
Power(n) Net

$O\left(log_2 \frac{1}{\sqrt{\varepsilon}}\right)$ neurons



$$y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$$

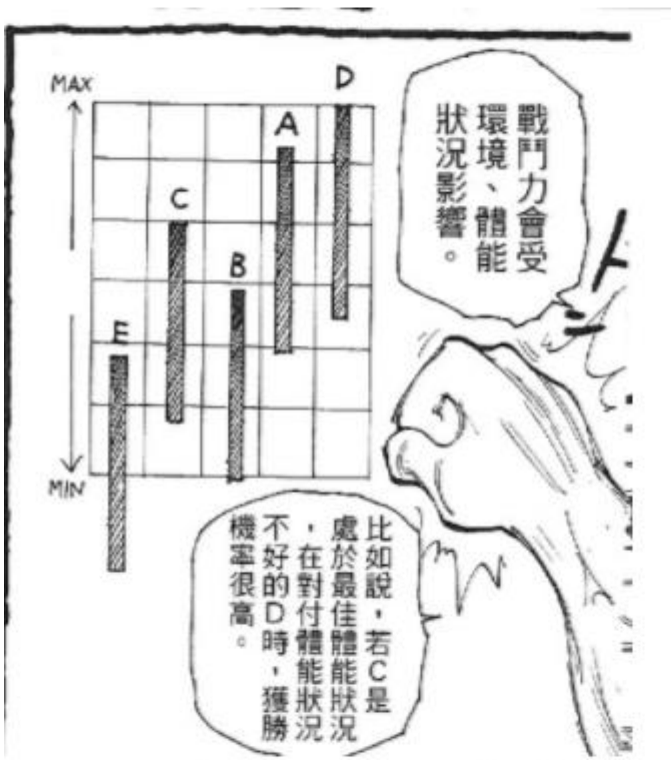
$O\left(log_2 \frac{1}{\sqrt{\varepsilon}}\right)$ neurons



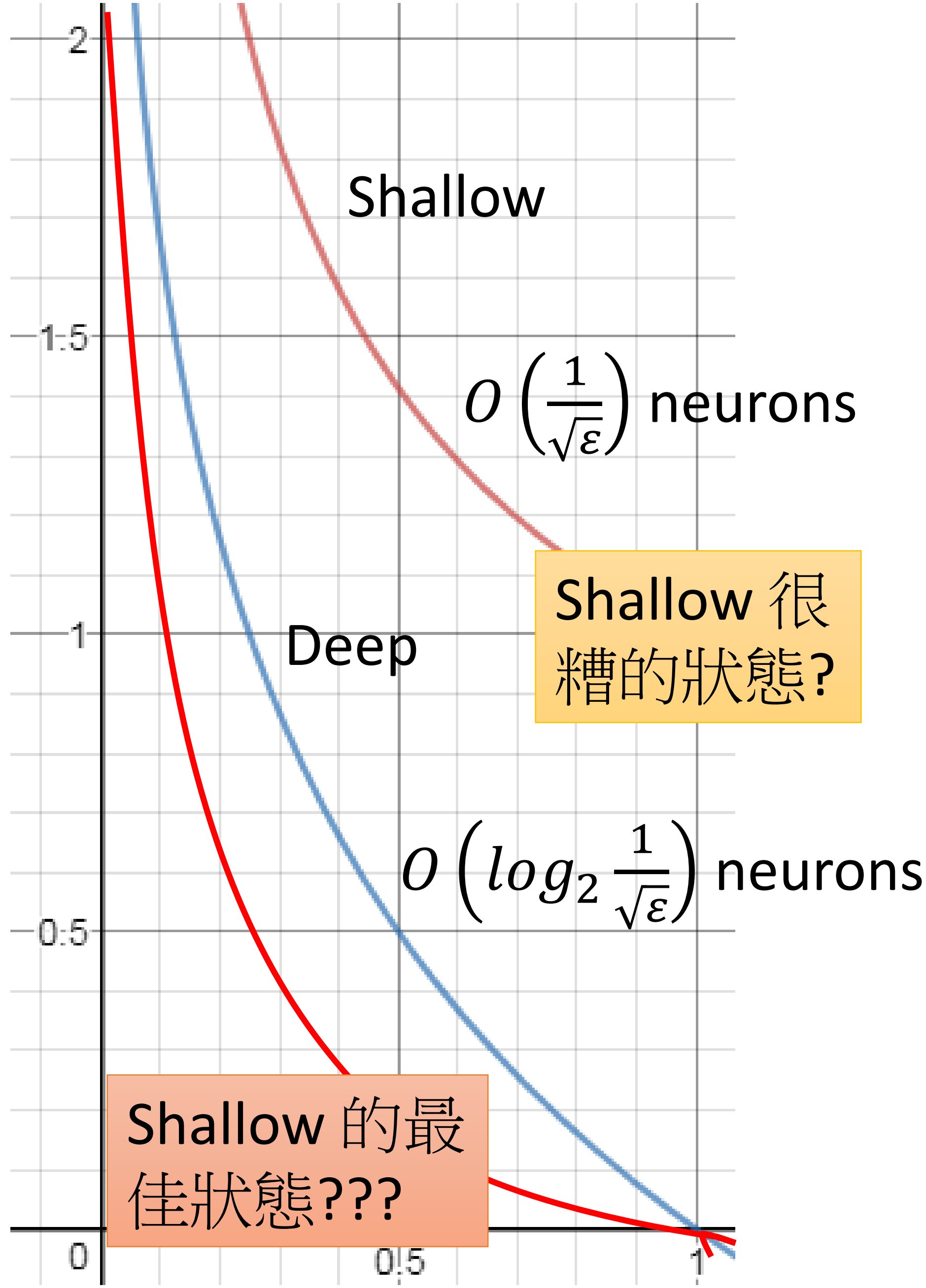
Use polynomial function to fit other functions.

Deep v.s. Shallow

This is not sufficient to show the power of deep.



(獵人第二十卷)



Is Deep better than Shallow?

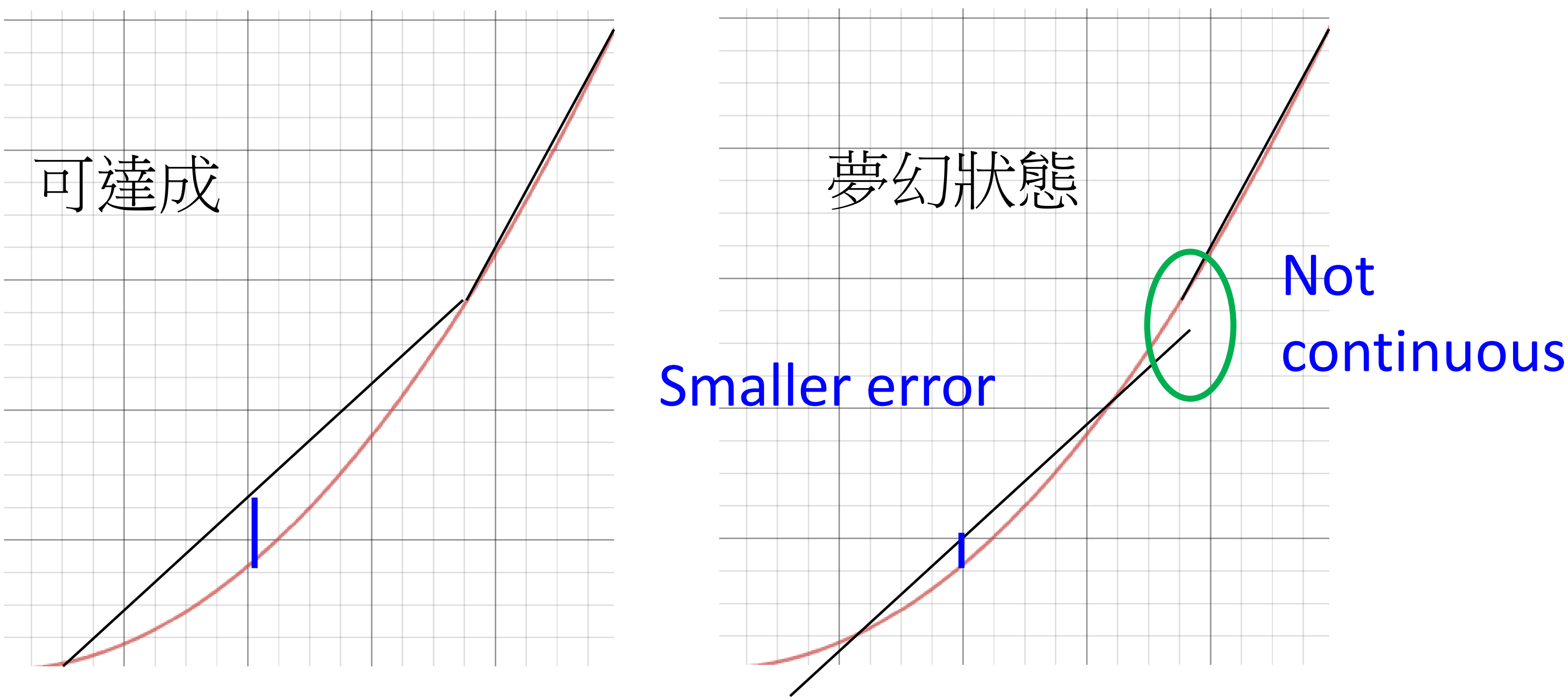
Best of Shallow

$$\max_{0\le x\le 1}|f(x)-f^*(x)|\le\varepsilon$$

$$\sqrt{\int_0^1|f(x)-f^*(x)|^2\,dx}\le\varepsilon$$

Use Euclidean

- A relu network is a piecewise linear function.
- Using the least pieces to fit the target function.



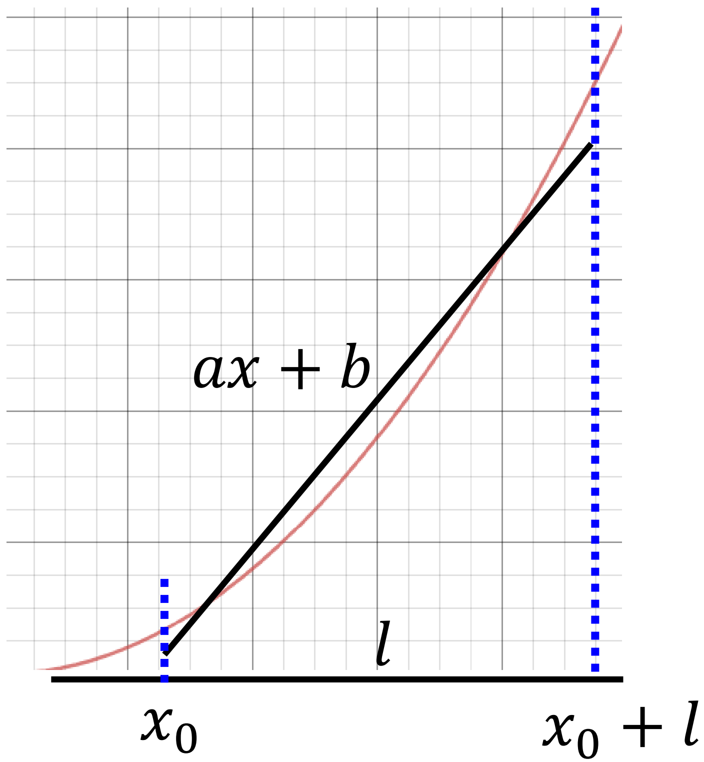
The lines do not have to connect the end points.

Best of Shallow

$$\sqrt{\int_0^1 |f(x) - f^*(x)|^2 \, dx} \leq \varepsilon$$

Use Euclidean

- Given a piece, what is the smallest error



$$e^2 = \int_{x_0}^{x_0+l} \left(x^2 - (ax + b)\right)^2 dx$$

Find a and b to minimize e^2

The minimum value of e^2 is $\frac{l^5}{180}$

Warning of Math

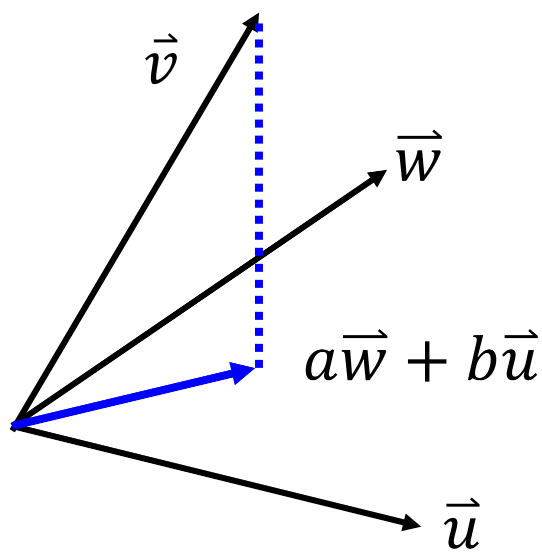
Intuition

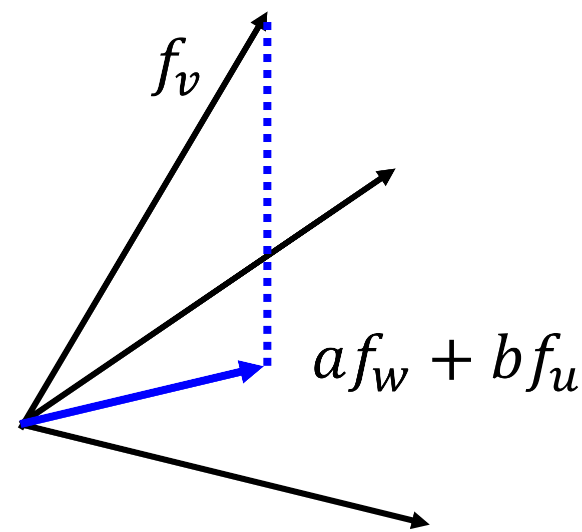
$$e^2 = \int_{x_0}^{x_0+l} \left(x^2 - (ax + b\ )\right)^2 dx$$

$$f_v = x^2 \qquad f_w = x \qquad f_u = 1$$

Minimize
$$\|\vec{v} - (a\vec{w} + b\vec{u})\|^2$$

Minimize
$$\|f_v - (af_w + bf_u)\|^2$$





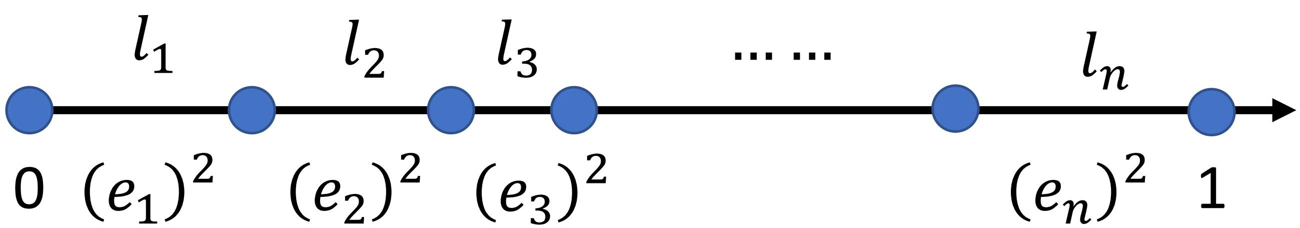
End of Warning

Best of Shallow

The minimum value of e² is $\frac{l^5}{180}$

- If you have n pieces, what is the best way to arrange the n pieces.

$$\sum_{i=1}^{n} l_i = 1$$



$$E^2 = \sum_{i=1}^{n} (e_i)^2 = \sum_{i=1}^{n} \frac{(l_i)^5}{180}$$

$$l_i = 1/n$$

The best way is "equal segment"

$$E^2 = \sum_{i=1}^{n} \frac{(1/n)^5}{180} = \frac{1}{180} \frac{1}{n^4}$$

Warning of Math

Hölder's inequality

$$\sum_{i=1}^{n} l_i = 1$$

Minimize $\sum_{i=1}^{n}(l_i)^5$

- Given $\{a_1, a_2, \cdots, a_n\}$ and $\{b_1, b_2, \cdots, b_n\}$

$$\sum_{i=1}^{n}|a_i b_i| \le \left(\sum_{i=1}^{n}|a_i|^p\right)^{1/p}\left(\sum_{i=1}^{n}|b_i|^q\right)^{1/q} \qquad \frac{1}{p}+\frac{1}{q}=1$$

$$1+\frac{p}{q}=p \qquad 1-p=-\frac{p}{q}$$

- Given $\{l_1, l_2, \cdots, l_n\}$ and $\{1,1,\cdots,1\}$

$$\underbrace{\boxed{\sum_{i=1}^{n} l_i}}_{=1} \le \left(\sum_{i=1}^{n} {l_i}^p\right)^{1/p}\left(\underbrace{\boxed{\sum_{i=1}^{n} 1^q}}_{=n}\right)^{1/q}$$

$$n^{-1/q} \le \left(\sum_{i=1}^{n} {l_i}^p\right)^{1/p}$$

$$n^{\overset{1-p}{\cancel{-p/q}}} \le \sum_{i=1}^{n} {l_i}^p$$

p=5

$$n^{-4} \le \sum_{i=1}^{n} {l_i}^5$$

End of Warning

Best of Shallow

The minimum value of e² is $\frac{l^5}{180}$

- If you have n pieces, what is the best way to arrange the n pieces.

$$E^2 = \frac{1}{180}\frac{1}{n^4} \Rightarrow E = \sqrt{\frac{1}{180}\frac{1}{n^2}}$$

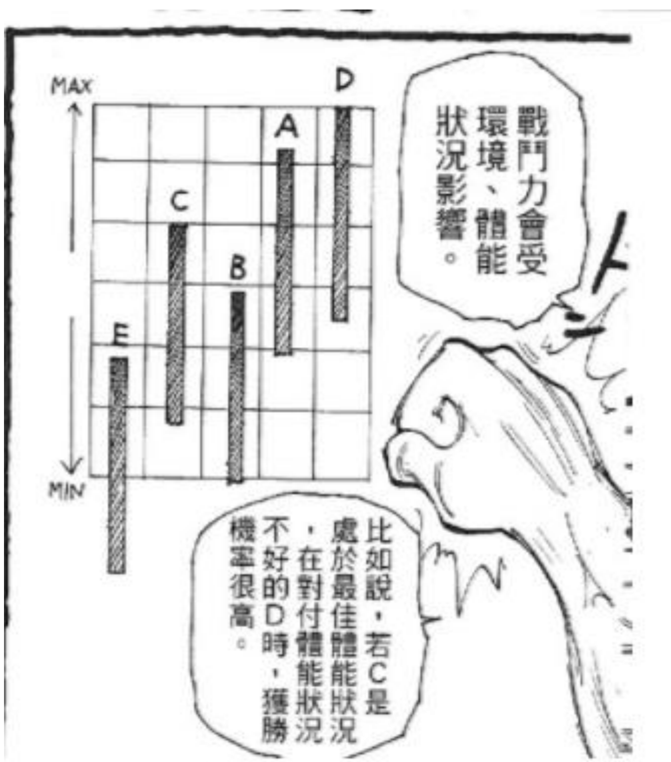
To make $E \leq \varepsilon$, what is the n we need?

$$E = \sqrt{\frac{1}{180}\frac{1}{n^2}} \leq \varepsilon \qquad n^2 \geq \sqrt{\frac{1}{180}}\frac{1}{\varepsilon} \qquad n \geq \sqrt[4]{\frac{1}{180}}\sqrt{\frac{1}{\varepsilon}}$$

At least $O\left(\frac{1}{\sqrt{\varepsilon}}\right)$ neurons

Deep v.s. Shallow

Deep is exponentially better than shallow.



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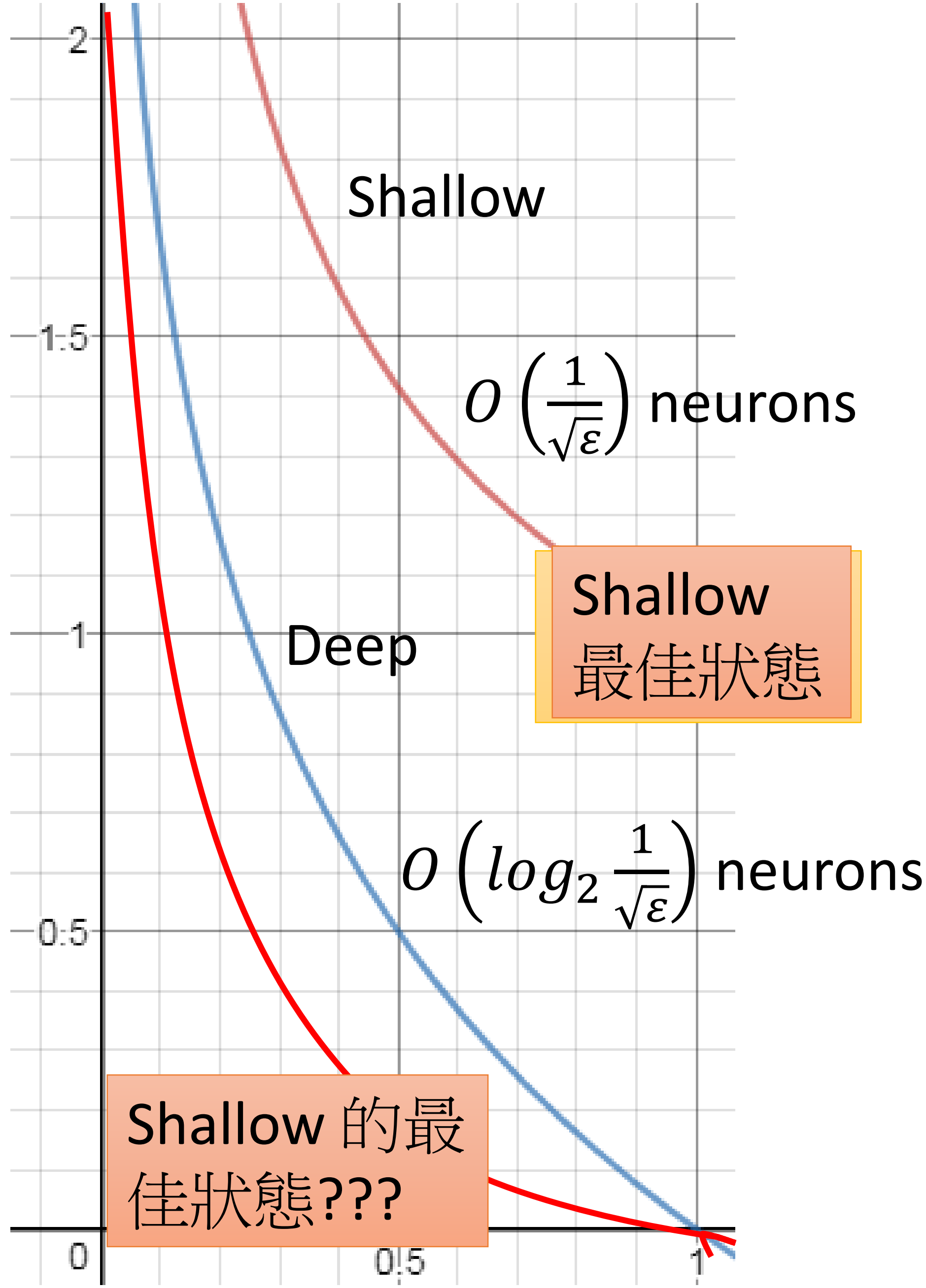
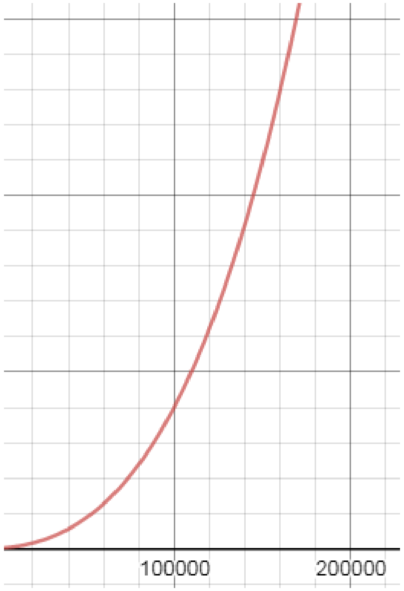


More related theories

More Theories

- A function expressible by a 3-layer feedforward network cannot be approximated by 2-layer network.
  - Unless the width of 2-layer network is VERY large
  - Applied on activation functions beyond relu



The width of 3-layer network is K.

The width of 2-layer network should be $Ae^{BK^{4/19}}$.

Ronen Eldan, Ohad Shamir, "The Power of Depth for Feedforward Neural Networks", COLT, 2016

More Theories

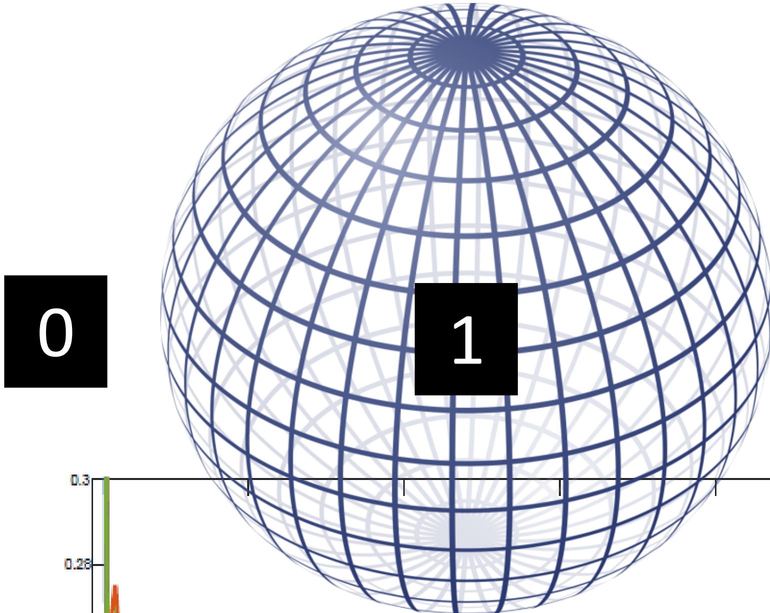
- A function expressible by a deep feedforward network cannot be approximated by a shallow network.
  - Unless the width of the shallow network is VERY large
  - Applied on activation functions beyond relu

Deep Network:

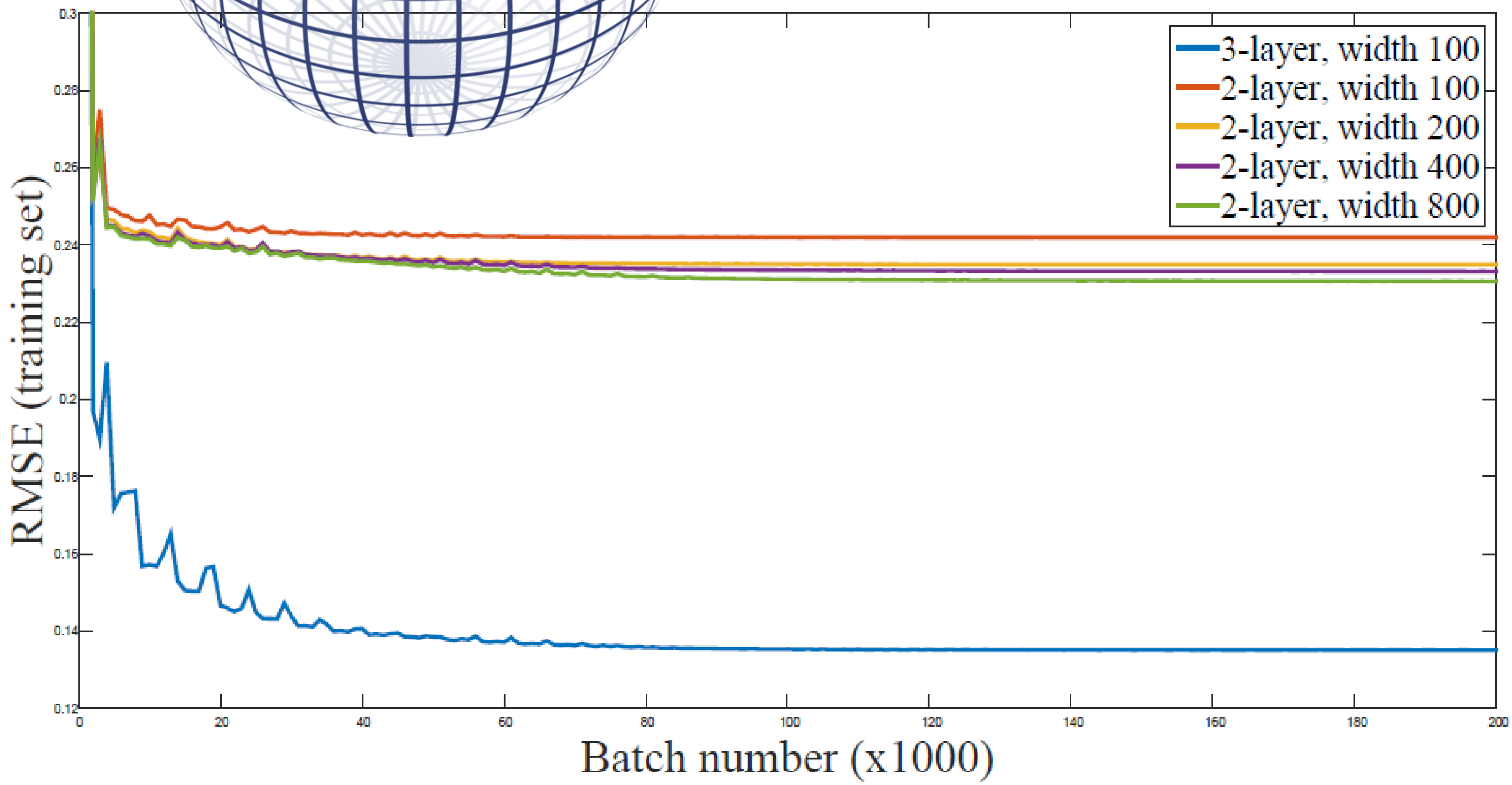
$\Theta(k^3)$ layers, $\Theta(1)$ nodes per layer, $\Theta(1)$ distinct parameters

Shallow Network: $\Theta(k)$ layers $\Rightarrow$ $\Omega(2^k)$ nodes

Matus Telgarsky, “Benefits of depth in neural networks”, COLT, 2016



Itay Safran, Ohad Shamir, “Depth-Width Tradeoffs in Approximating Natural Functions with Neural Networks”, ICML, 2017



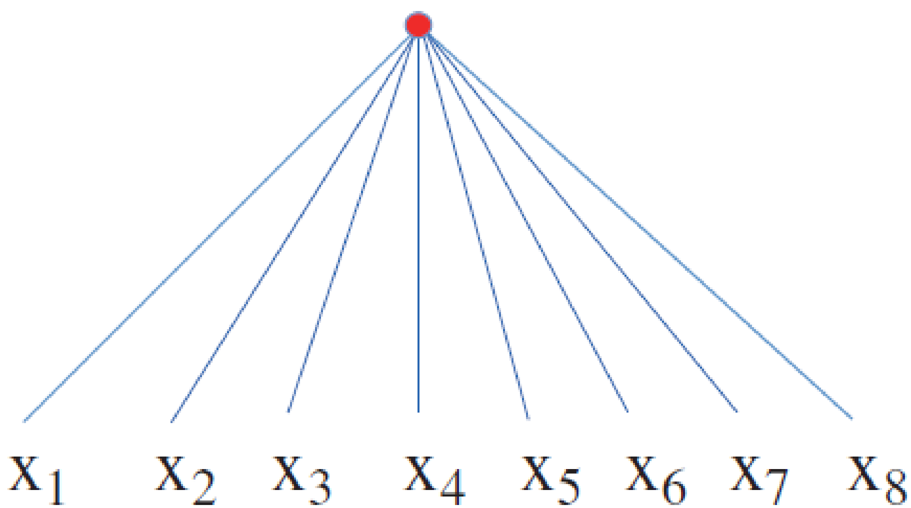
More Theories

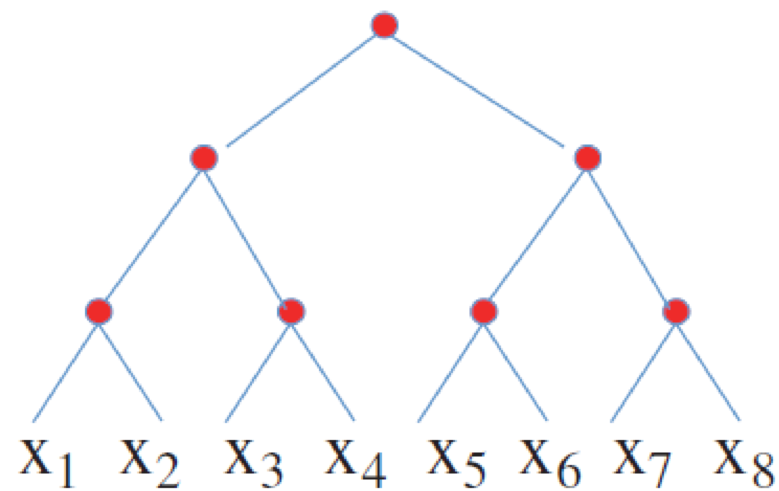
Dmitry Yarotsky, “Error bounds for approximations with deep ReLU networks”, arXiv, 2016
Dmitry Yarotsky, “Optimal approximation of continuous functions by very deep ReLU networks”, arXiv 2018
Shiyu Liang, R. Srikant, “Why Deep Neural Networks for Function Approximation?”, ICLR, 2017
Itay Safran, Ohad Shamir, “Depth-Width Tradeoffs in Approximating Natural Functions with Neural Networks”, ICML, 2017

If a function f has “certain degree of complexity”

- Approximating f to accuracy ε in the L2 norm using a fixed depth ReLU network requires at least $poly(1/\varepsilon)$
- There exist a ReLU network of depth and width at most $poly\big(log(1/\varepsilon)\big)$ that can achieve the approximation.

The Nature of Functions





Hrushikesh Mhaskar, Qianli Liao, Tomaso Poggio, When and Why Are Deep Networks Better Than Shallow Ones?, AAAI, 2017

Concluding Remarks