Regression

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Regression: Output a scalar

- Stock Market Forecast

$f($  $) =$ Dow Jones Industrial Average at tomorrow

- Self-driving Car

$f($  $) =$ 方向盤角度

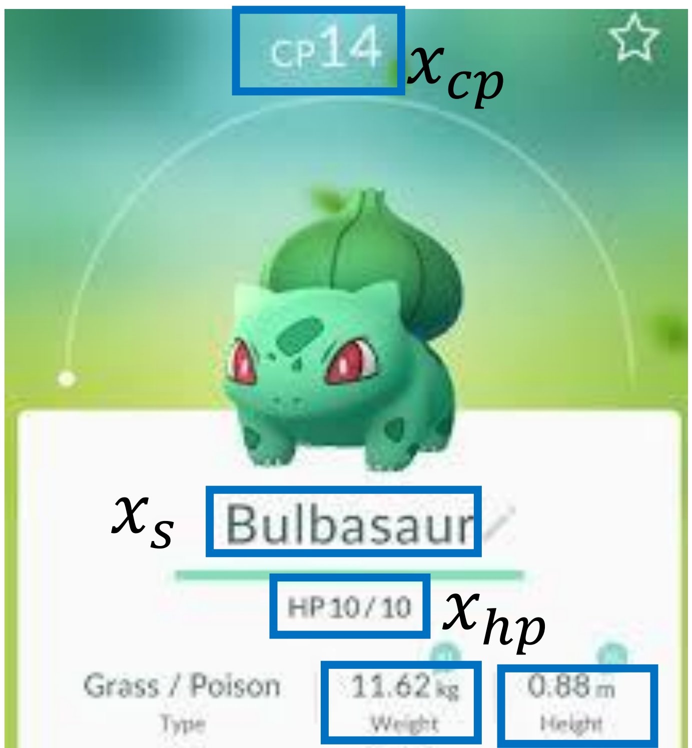
- Recommendation

$f($ 使用者 A 商品 B $) =$ 購買可能性

Example Application

- Estimating the Combat Power (CP) of a pokemon after evolution

$f($



x_{cp} x_s x_{hp} x_w x_h x

$) =$ CP after evolution y

Step 1: Model

$y = b + w \cdot x_{cp}$

A set of function

Model

$f_1, f_2 \cdots$

w and b are parameters (can be any value)

f_1: $y = 10.0 + 9.0 \cdot x_{cp}$

f_2: $y = 9.8 + 9.2 \cdot x_{cp}$

f_3: $y = -0.8 - 1.2 \cdot x_{cp}$

…… infinite

$f($ 

$x) =$ CP after evolution y

Linear model: $y = b + \sum w_i x_i$

x_i: $x_{cp}, x_{hp}, x_w, x_h \ldots$

feature

w_i: weight, b: bias

Step 2: Goodness of Function

$y = b + w \cdot x_{cp}$

A set of function

Model

$f_1, f_2 \cdots$

Training Data

function input:

function Output (scalar):

Step 2: Goodness of Function

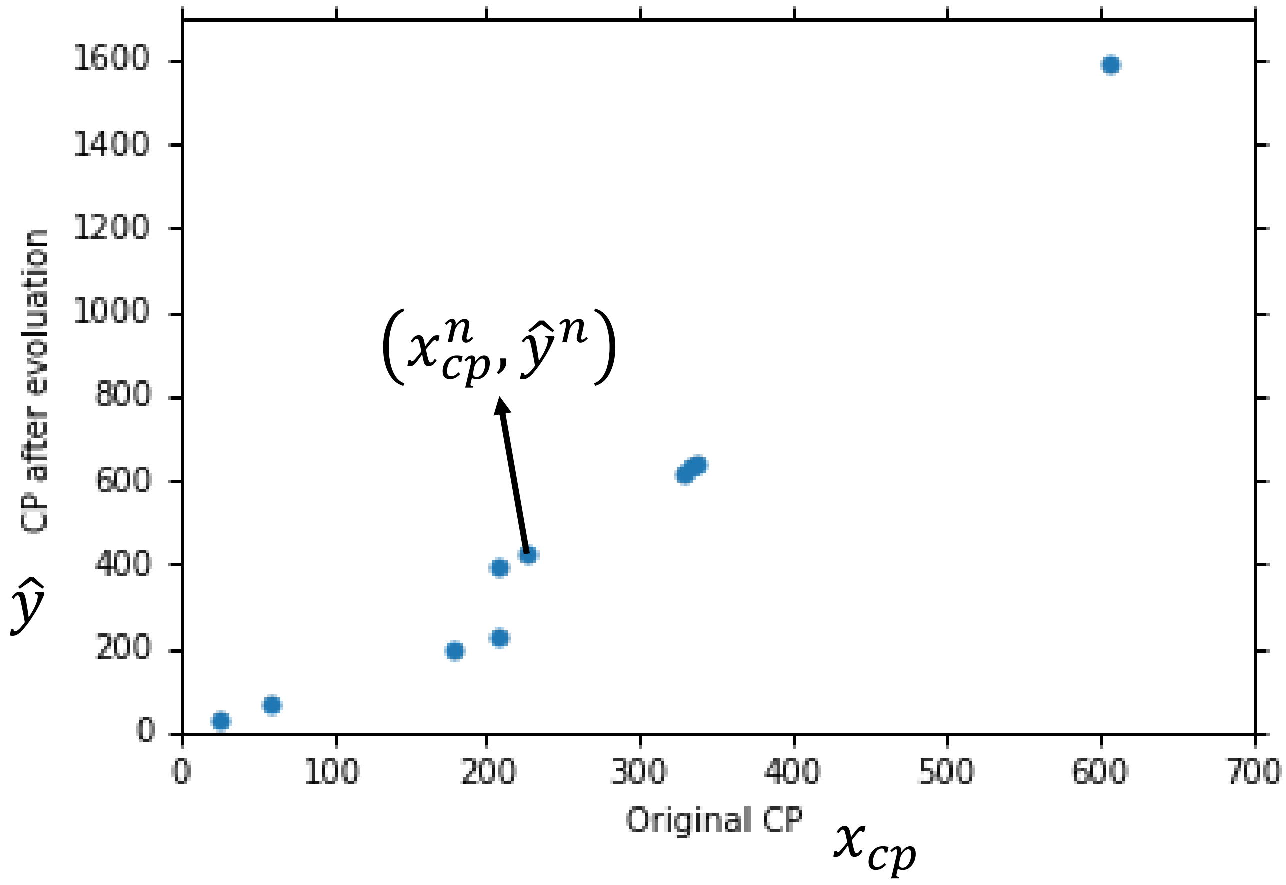
Training Data:
10 pokemons

$(x^1, \hat{y}^1)$

$(x^2, \hat{y}^2)$

⋮

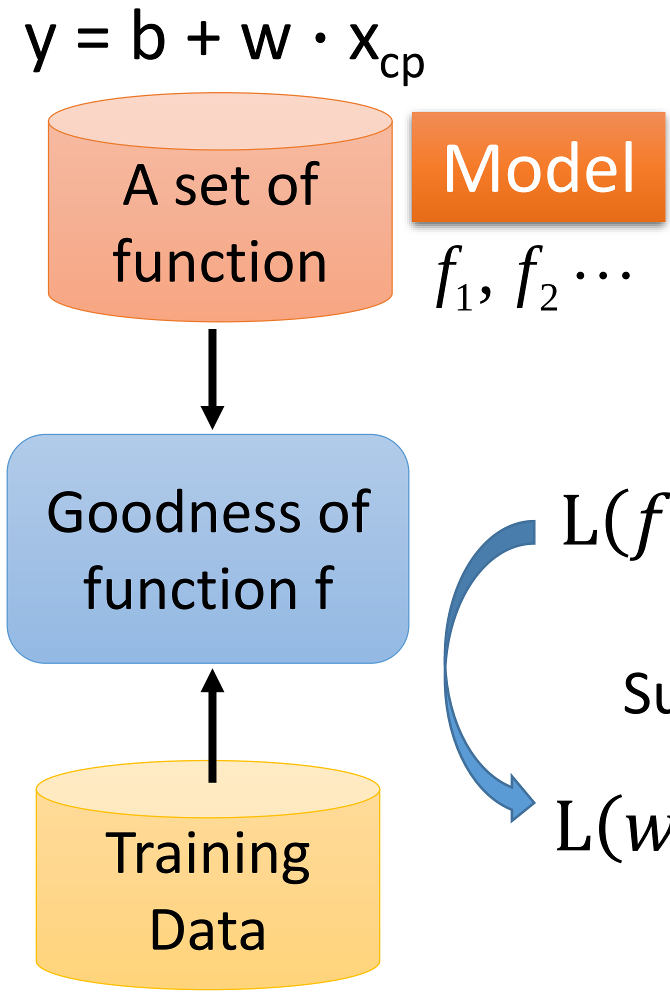
$(x^{10}, \hat{y}^{10})$



This is real data.

Source: https://www.openintro.org/stat/data/?data=pokemon

Step 2: Goodness of Function



Loss function L:

Input: a function, output: how bad it is

Estimation error

$$L(f) = \sum_{n=1}^{10} \left(\hat{y}^n - f\left(x_{cp}^n\right) \right)^2$$

Sum over examples

Estimated y based on input function

$$L(w, b) = \sum_{n=1}^{10} \left(\hat{y}^n - \left(b + w \cdot x_{cp}^n\right) \right)^2$$

Step 2: Goodness of Function

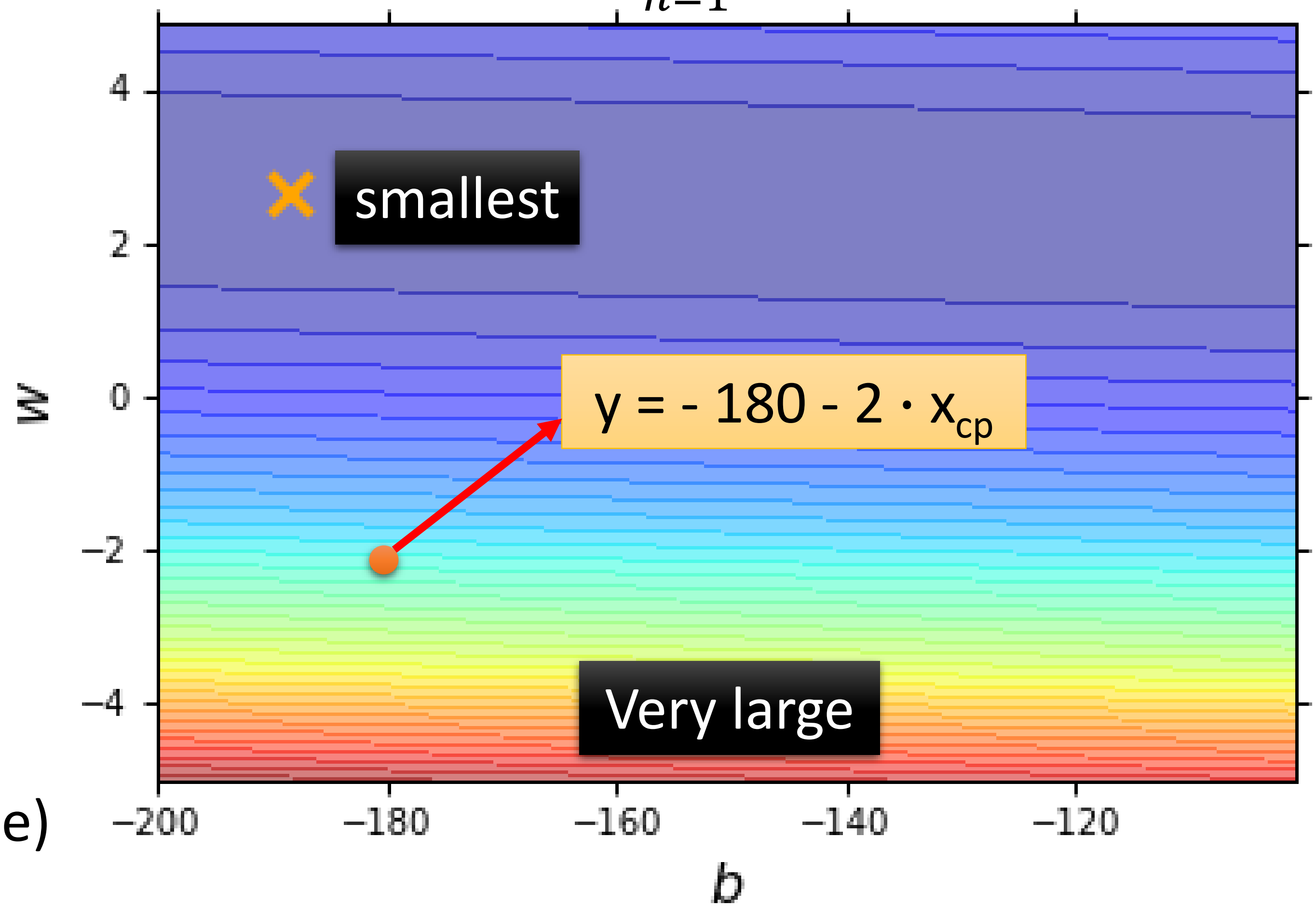
- Loss Function

$$L(w,b) = \sum_{n=1}^{10} \left(\hat{y}^n - \left(b + w \cdot x_{cp}^n \right) \right)^2$$

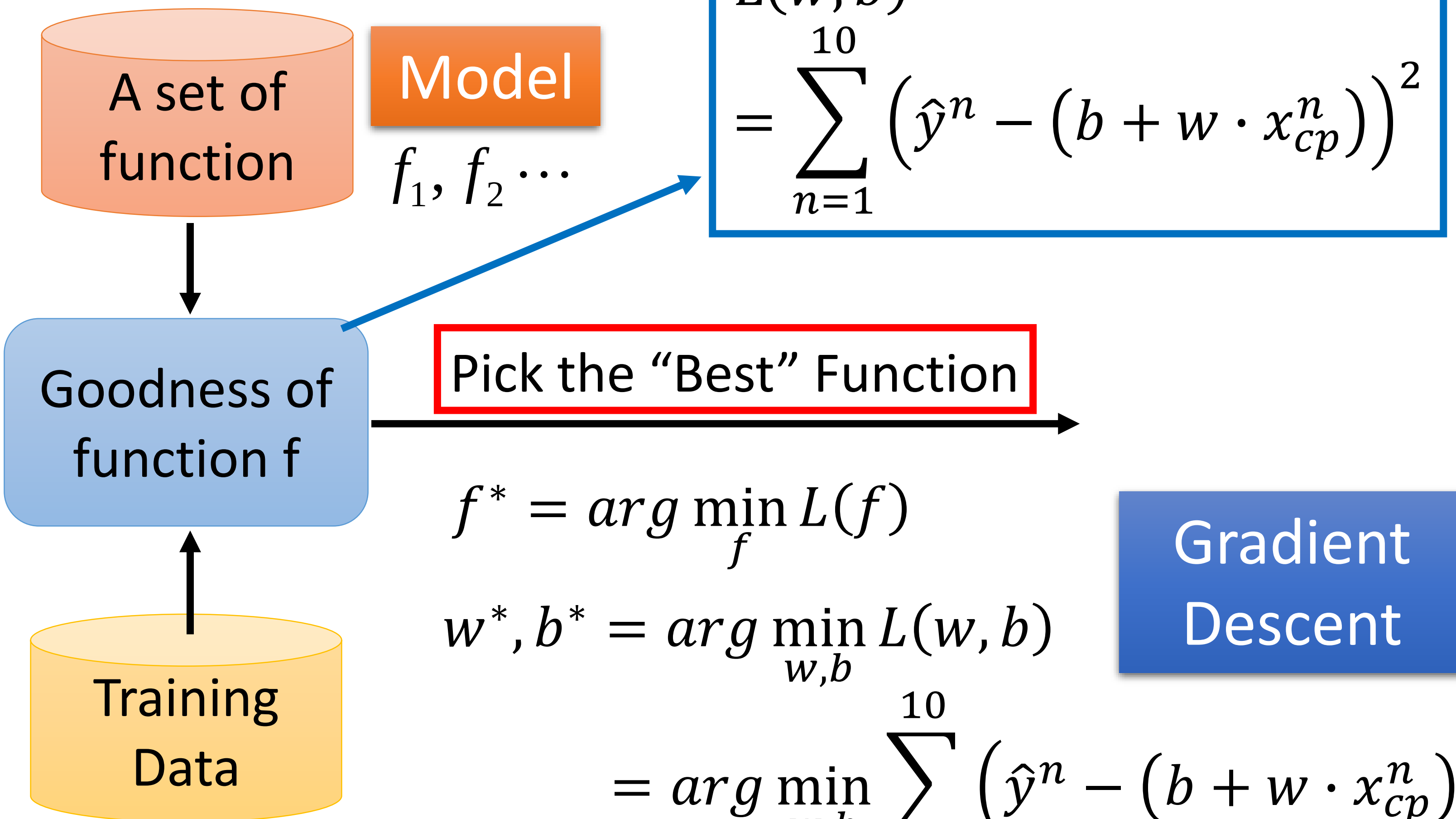
Each point in the figure is a function

The color represents $L(w,b)$.

(true example)



Step 3: Best Function



$f_1, f_2 \cdots$

$$\mathrm{L}(w,b) = \sum_{n=1}^{10}\left(\hat{y}^n - (b + w \cdot x_{cp}^n)\right)^2$$

$$f^* = arg\min_f L(f)$$

$$w^*, b^* = arg\min_{w,b} L(w,b)$$

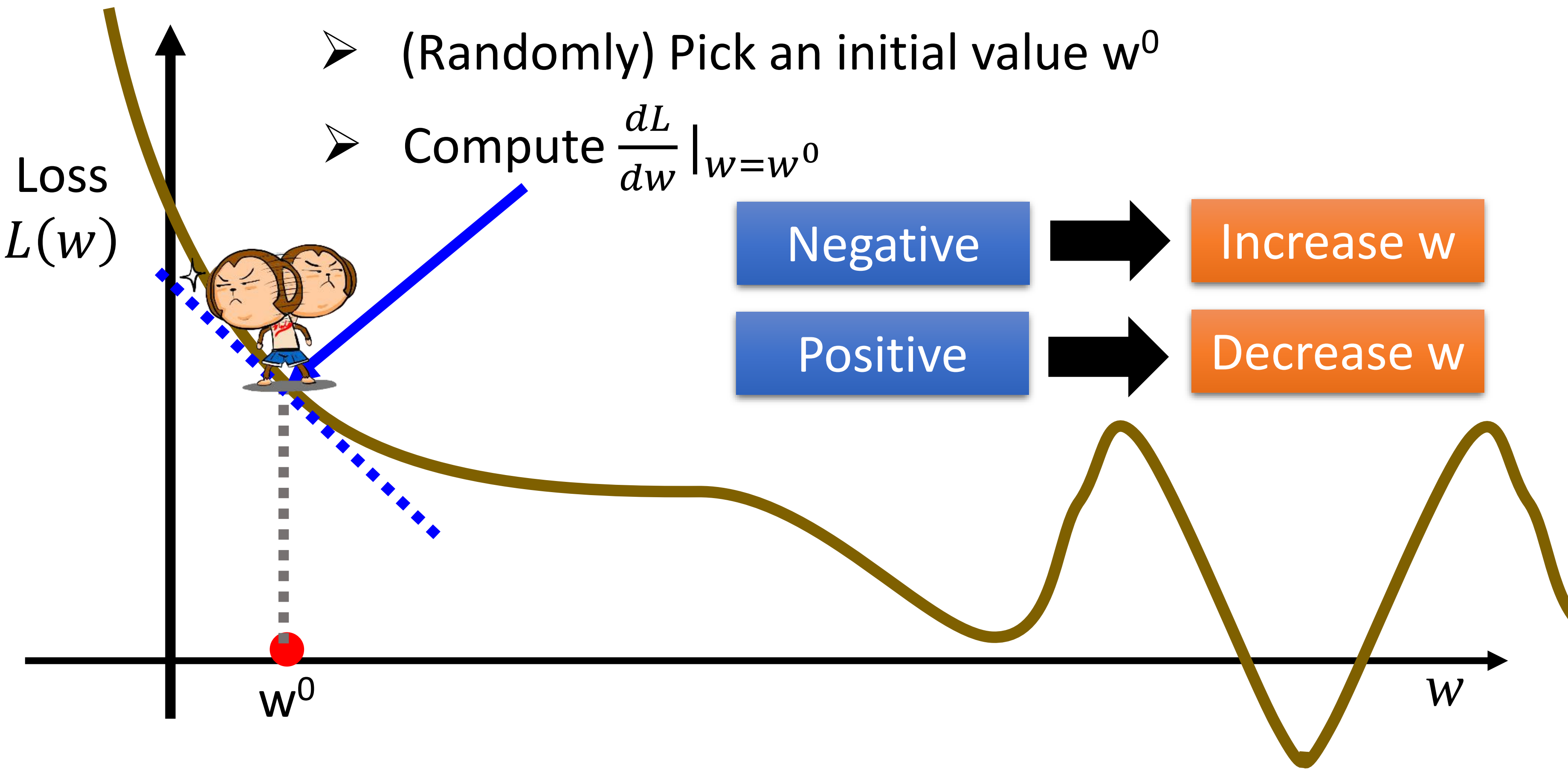
$$= arg\min_{w,b} \sum_{n=1}^{10}\left(\hat{y}^n - (b + w \cdot x_{cp}^n)\right)^2$$

Step 3: Gradient Descent

$$w^* = arg \min_w L(w)$$

- Consider loss function $L(w)$ with one parameter w:
  - (Randomly) Pick an initial value w^0
  - Compute $\frac{dL}{dw}|_{w=w^0}$

Negative	➡	Increase w
Positive	➡	Decrease w



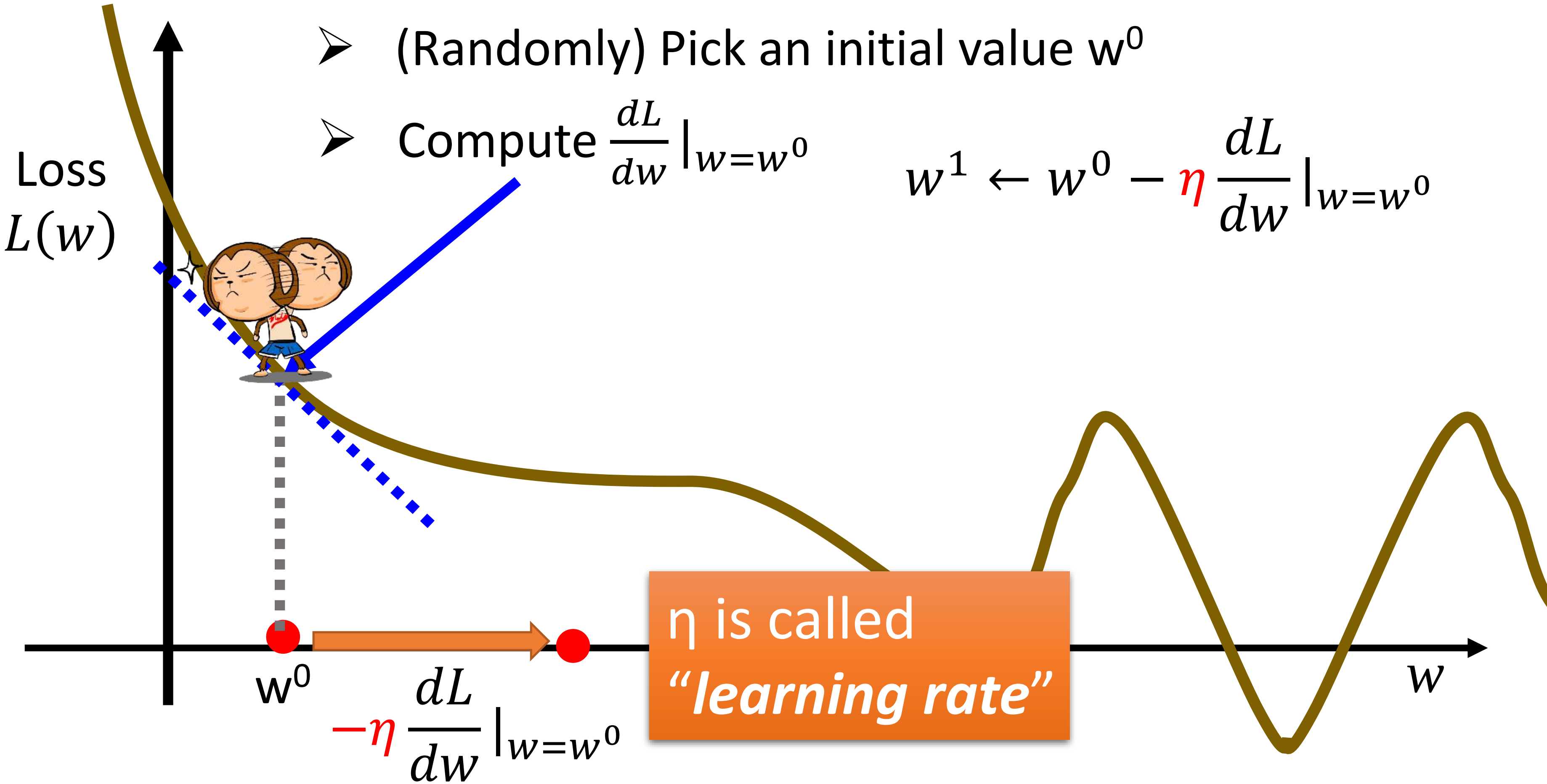
http://chico386.pixnet.net/album/photo/171572850

Step 3: Gradient Descent

$$w^* = arg \min_w L(w)$$

- Consider loss function $L(w)$ with one parameter w:
  - (Randomly) Pick an initial value w^0
  - Compute $\frac{dL}{dw}|_{w=w^0}$

$$w^1 \leftarrow w^0 - \eta \frac{dL}{dw}|_{w=w^0}$$



η is called *"learning rate"*

Step 3: Gradient Descent

$$w^* = arg \min_w L(w)$$

- Consider loss function $L(w)$ with one parameter w:

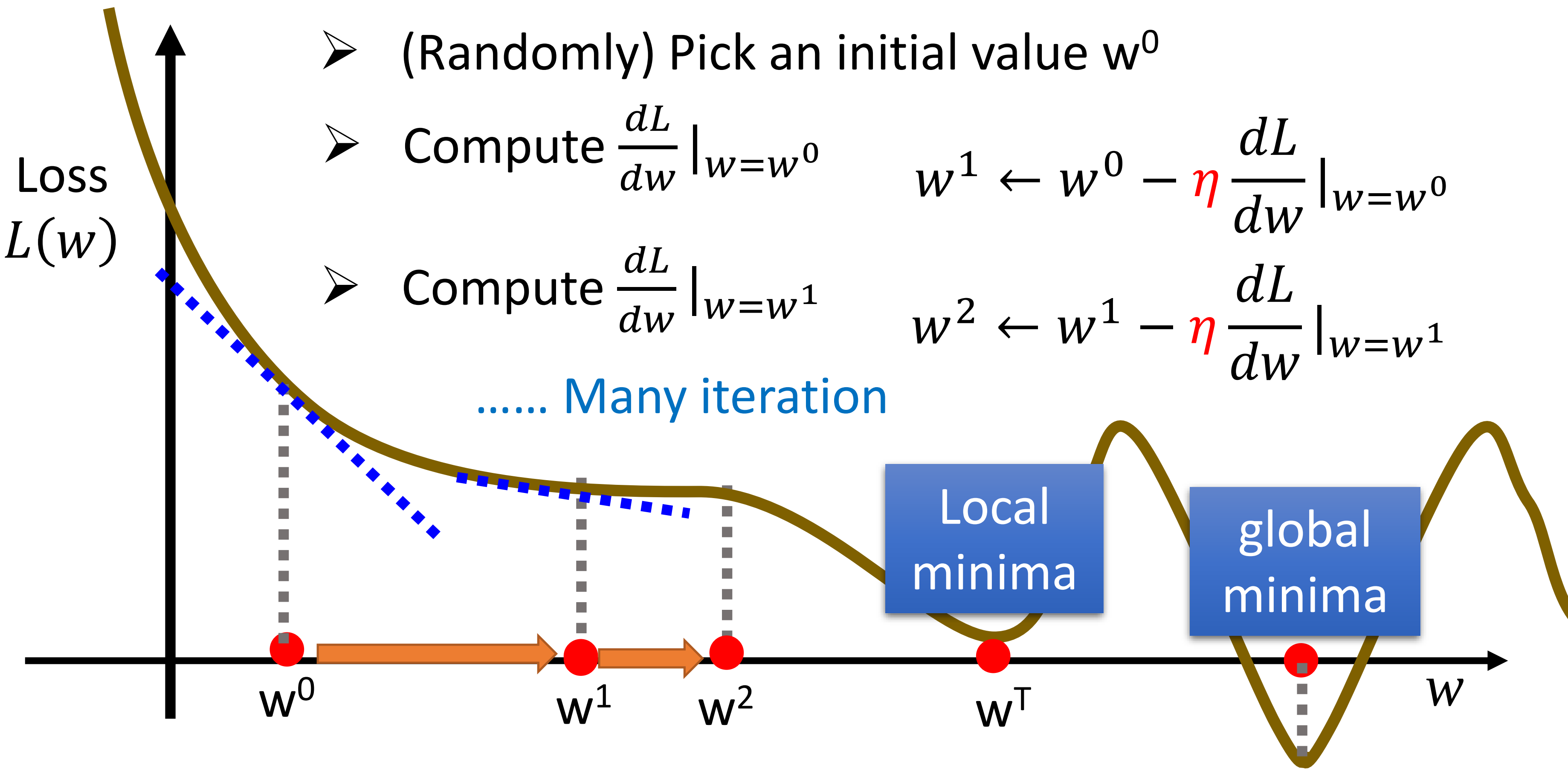
  - (Randomly) Pick an initial value w^0
  - Compute $\frac{dL}{dw}|_{w=w^0}$

    $$w^1 \leftarrow w^0 - \eta \frac{dL}{dw}|_{w=w^0}$$

  - Compute $\frac{dL}{dw}|_{w=w^1}$

    $$w^2 \leftarrow w^1 - \eta \frac{dL}{dw}|_{w=w^1}$$

…… Many iteration



Step 3: Gradient Descent

$$\begin{bmatrix} \frac{\partial L}{\partial w} \\ \frac{\partial L}{\partial b} \end{bmatrix} \text{gradient}$$

- How about two parameters? $w^*, b^* = arg \min_{w,b} L(w, b)$

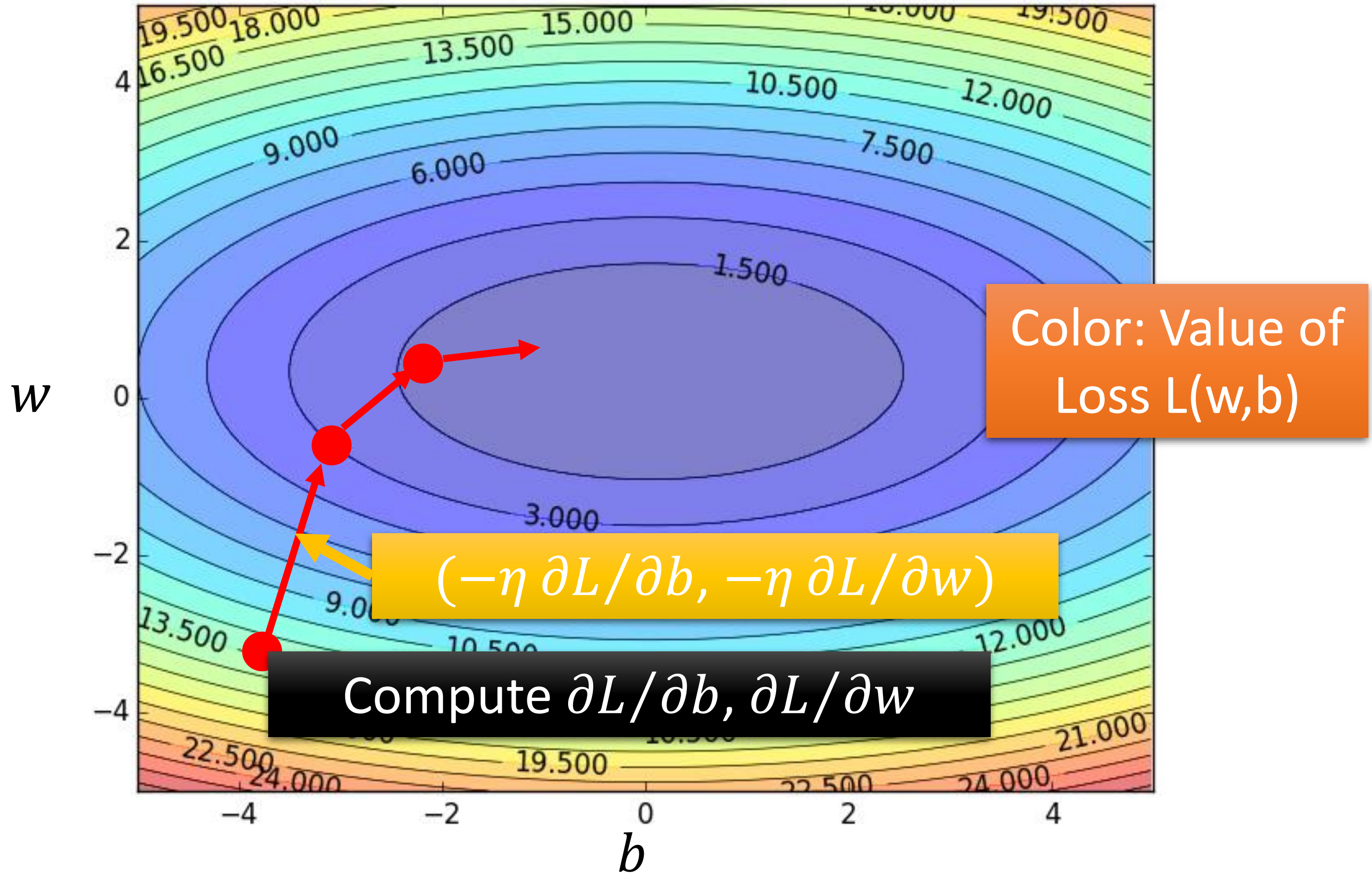
  - (Randomly) Pick an initial value w^0, b^0
  - Compute $\frac{\partial L}{\partial w}|_{w=w^0,b=b^0}$, $\frac{\partial L}{\partial b}|_{w=w^0,b=b^0}$

$$w^1 \leftarrow w^0 - \eta \frac{\partial L}{\partial w}|_{w=w^0,b=b^0} \qquad b^1 \leftarrow b^0 - \eta \frac{\partial L}{\partial b}|_{w=w^0,b=b^0}$$

  - Compute $\frac{\partial L}{\partial w}|_{w=w^1,b=b^1}$, $\frac{\partial L}{\partial b}|_{w=w^1,b=b^1}$

$$w^2 \leftarrow w^1 - \eta \frac{\partial L}{\partial w}|_{w=w^1,b=b^1} \qquad b^2 \leftarrow b^1 - \eta \frac{\partial L}{\partial b}|_{w=w^1,b=b^1}$$

Step 3: Gradient Descent



Step 3: Gradient Descent

- When solving:

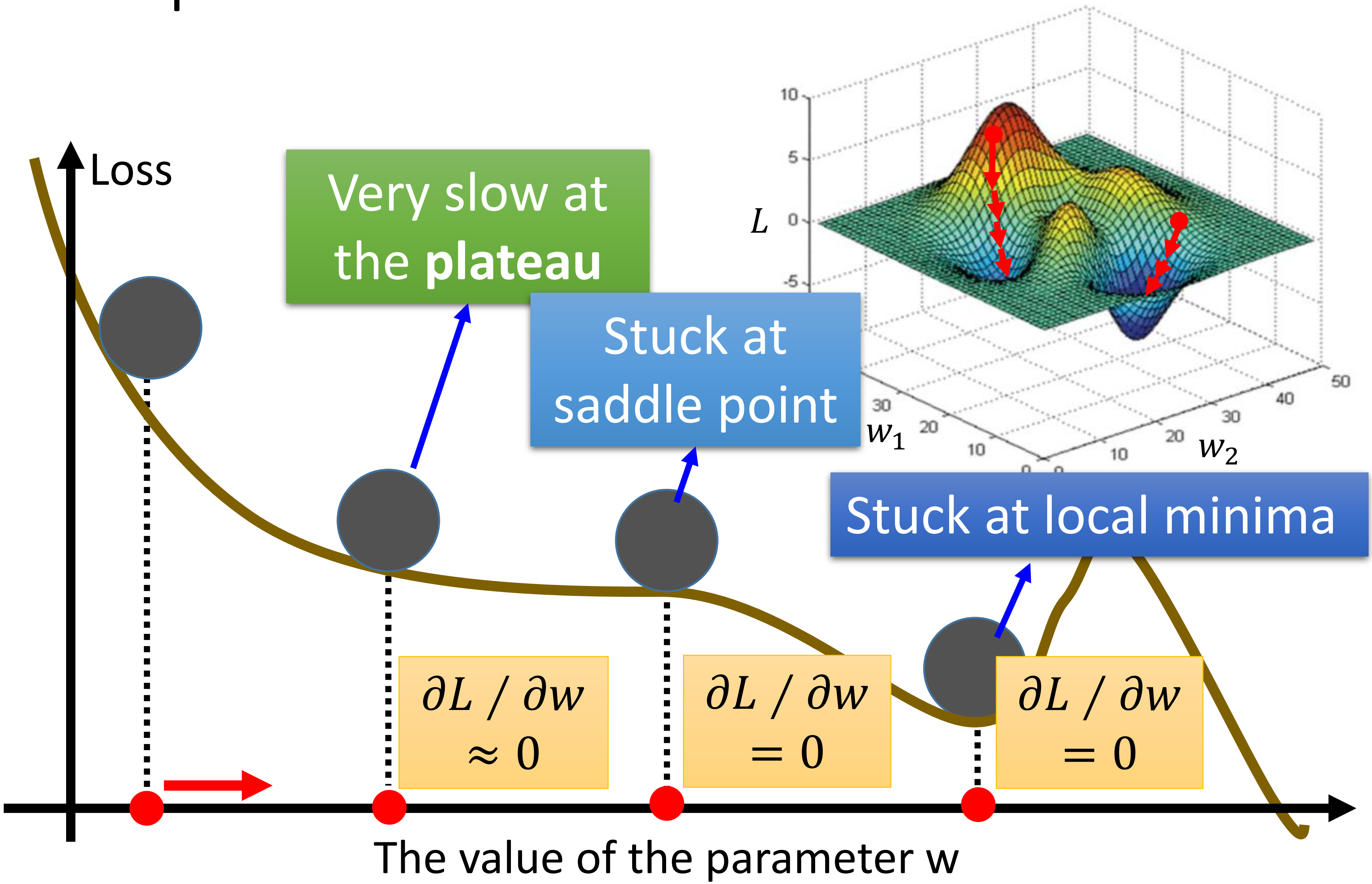
$$\theta^* = \arg\max_{\theta} L(\theta) \quad \text{by gradient descent}$$

- Each time we update the parameters, we obtain θ that makes $L(\theta)$ smaller.

$$L(\theta^0) > L(\theta^1) > L(\theta^2) > \cdots$$

Is this statement correct?

Step 3: Gradient Descent



Step 3: Gradient Descent

- Formulation of $\partial L/\partial w$ and $\partial L/\partial b$

$$L(w,b) = \sum_{n=1}^{10} \left(\hat{y}^n - \left(b + w \cdot x_{cp}^n\right)\right)^2$$

$$\frac{\partial L}{\partial w} =? \sum_{n=1}^{10} 2\left(\hat{y}^n - \left(b + w \cdot x_{cp}^n\right)\right)$$

$$\frac{\partial L}{\partial b} =?$$

Step 3: Gradient Descent

- Formulation of $\partial L / \partial w$ and $\partial L / \partial b$

$$L(w, b) = \sum_{n=1}^{10} \left(\hat{y}^n - \left(b + w \cdot x_{cp}^n \right) \right)^2$$

$$\frac{\partial L}{\partial w} =? \sum_{n=1}^{10} 2 \left(\hat{y}^n - \left(b + w \cdot x_{cp}^n \right) \right) \left(-x_{cp}^n \right)$$

$$\frac{\partial L}{\partial b} =? \sum_{n=1}^{10} 2 \left(\hat{y}^n - \left(b + w \cdot x_{cp}^n \right) \right)$$

Step 3: Gradient Descent

How's the results?

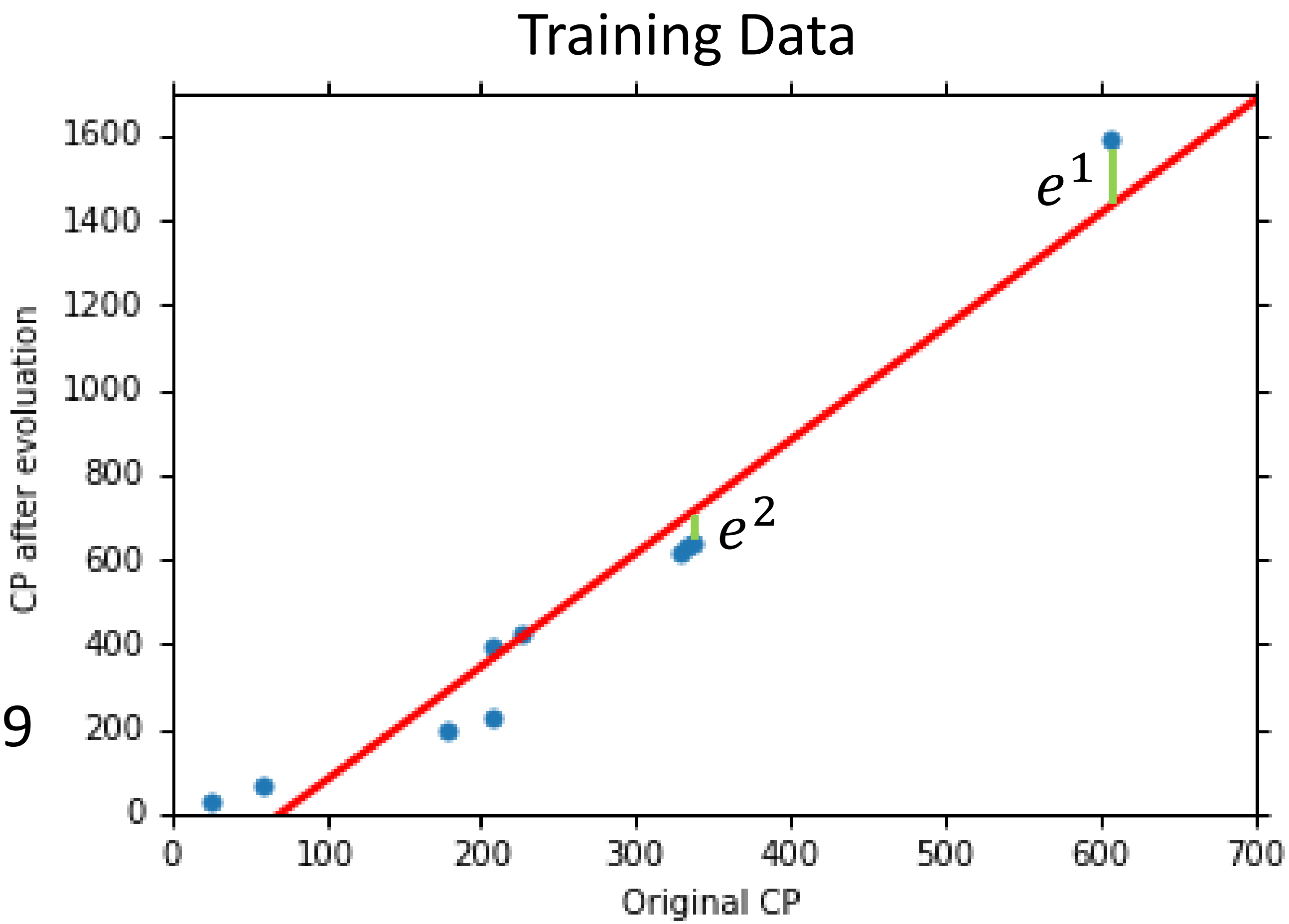
$y = b + w \cdot x_{cp}$

b = -188.4

w = 2.7

Average Error on Training Data

$$= \frac{1}{10}\sum_{n=1}^{10} e^n = 31.9$$



How's the results? - Generalization

What we really care about is the error on new data (testing data)

$y = b + w \cdot x_{cp}$

b = -188.4

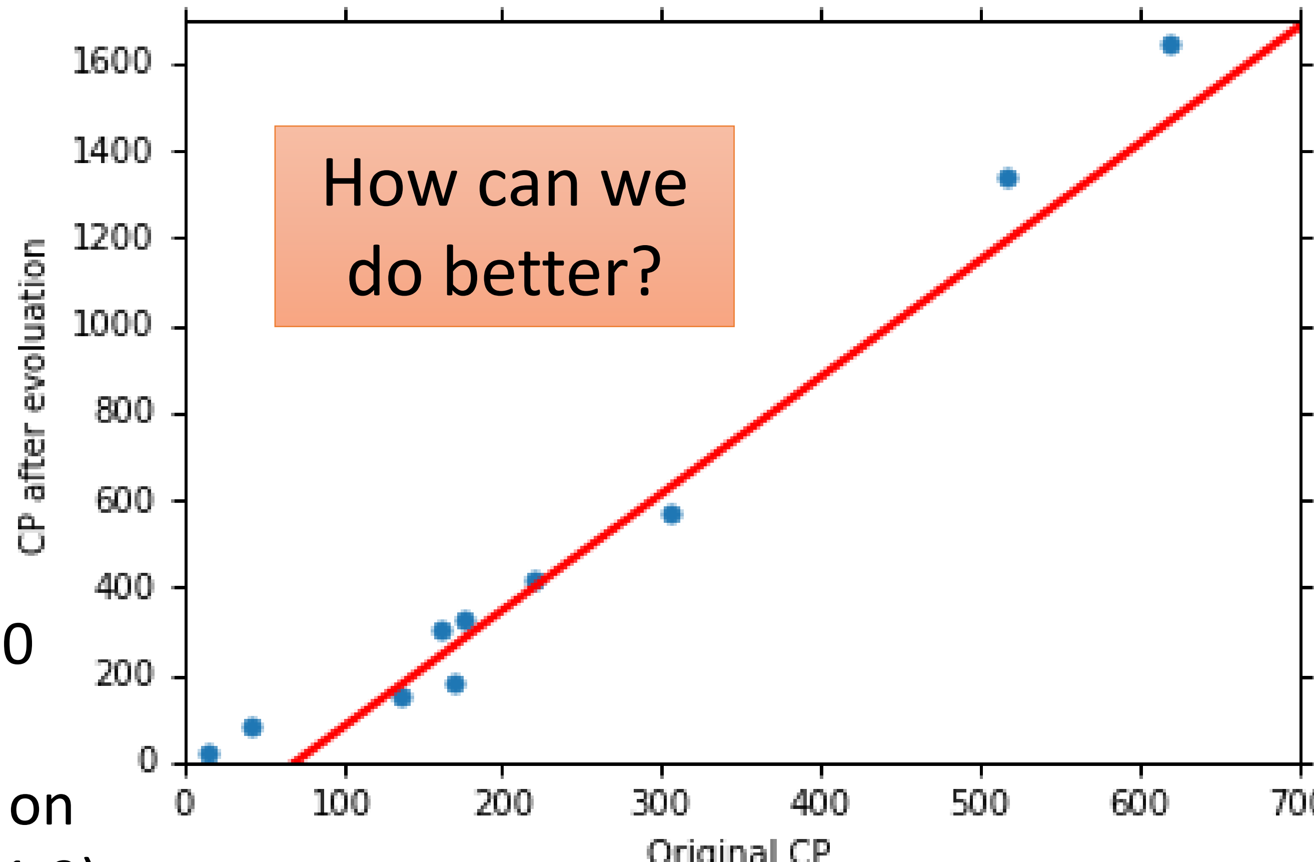
w = 2.7

Average Error on Testing Data

$$= \frac{1}{10}\sum_{n=1}^{10} e^n = 35.0$$

> Average Error on Training Data (31.9)

Another 10 pokemons as testing data



How can we do better?

Selecting another Model

$$y = b + w_1 \cdot x_{cp} + w_2 \cdot (x_{cp})^2$$

Best Function

$b = -10.3$

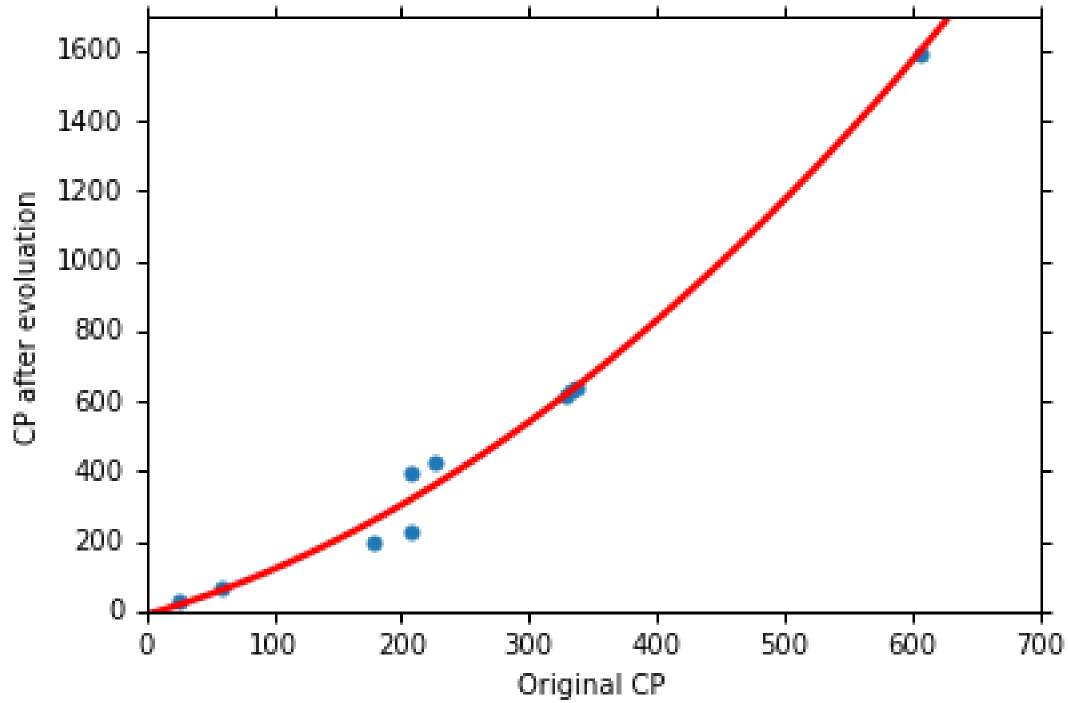
$w_1 = 1.0$, $w_2 = 2.7 \times 10^{-3}$

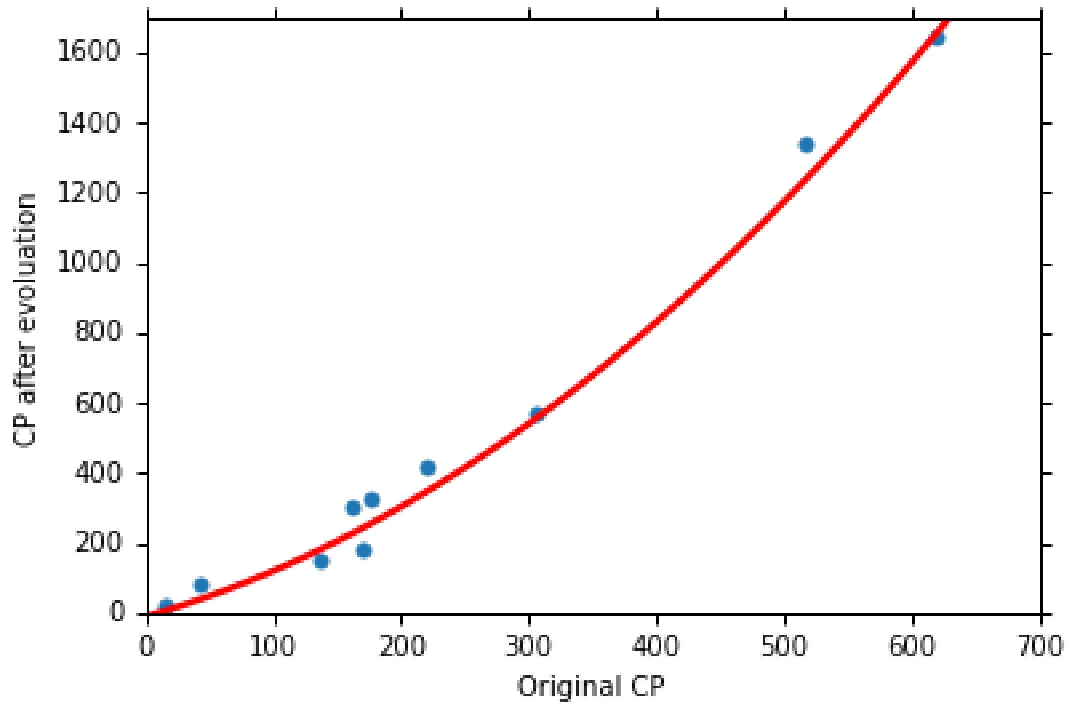
Average Error = 15.4

Testing:

Average Error = 18.4

Better! Could it be even better?





Selecting another Model

$$y = b + w_1 \cdot x_{cp} + w_2 \cdot (x_{cp})^2 + w_3 \cdot (x_{cp})^3$$

Best Function

$b = 6.4$, $w_1 = 0.66$

$w_2 = 4.3 \times 10^{-3}$

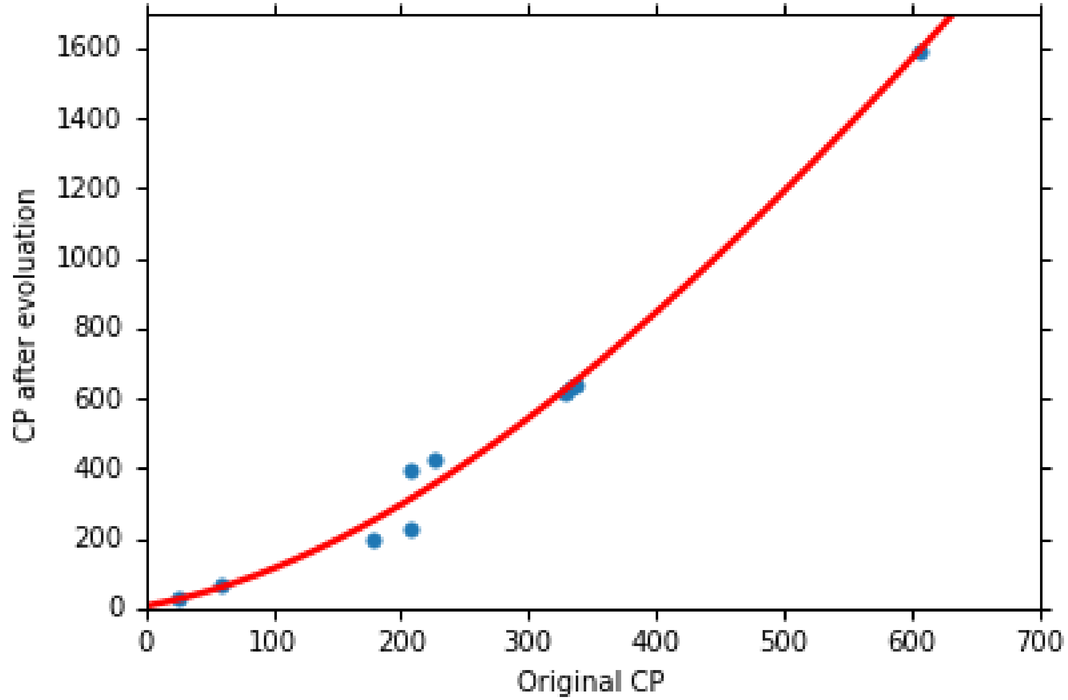
$w_3 = -1.8 \times 10^{-6}$

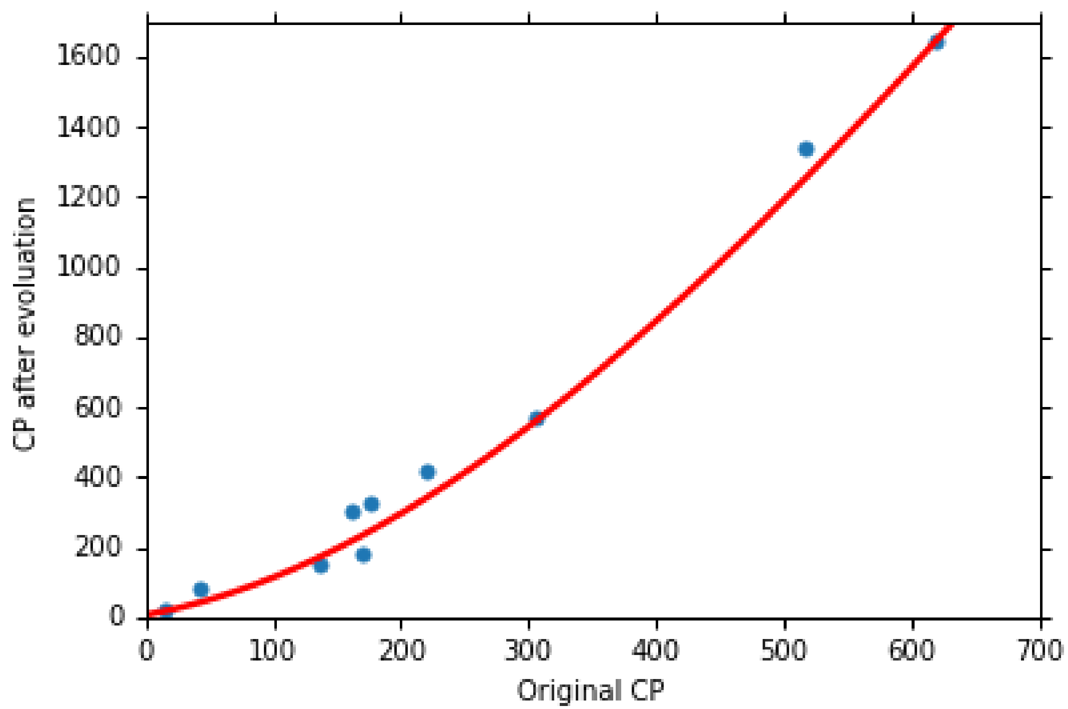
Average Error = 15.3

Testing:

Average Error = 18.1

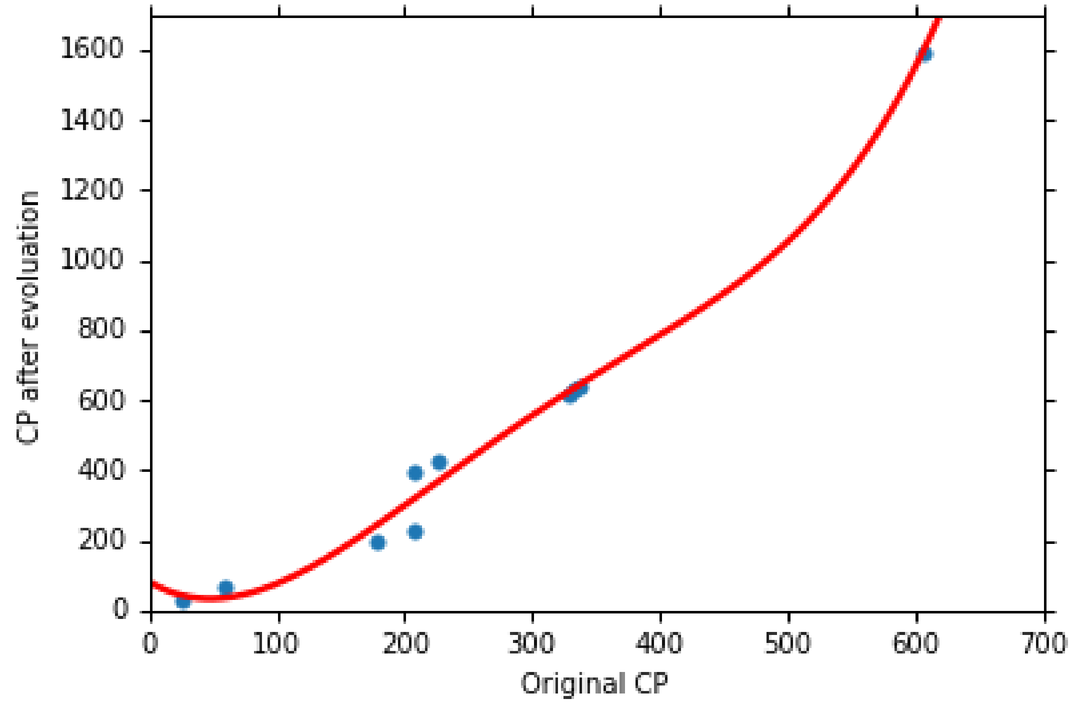
Slightly better.
How about more complex model?





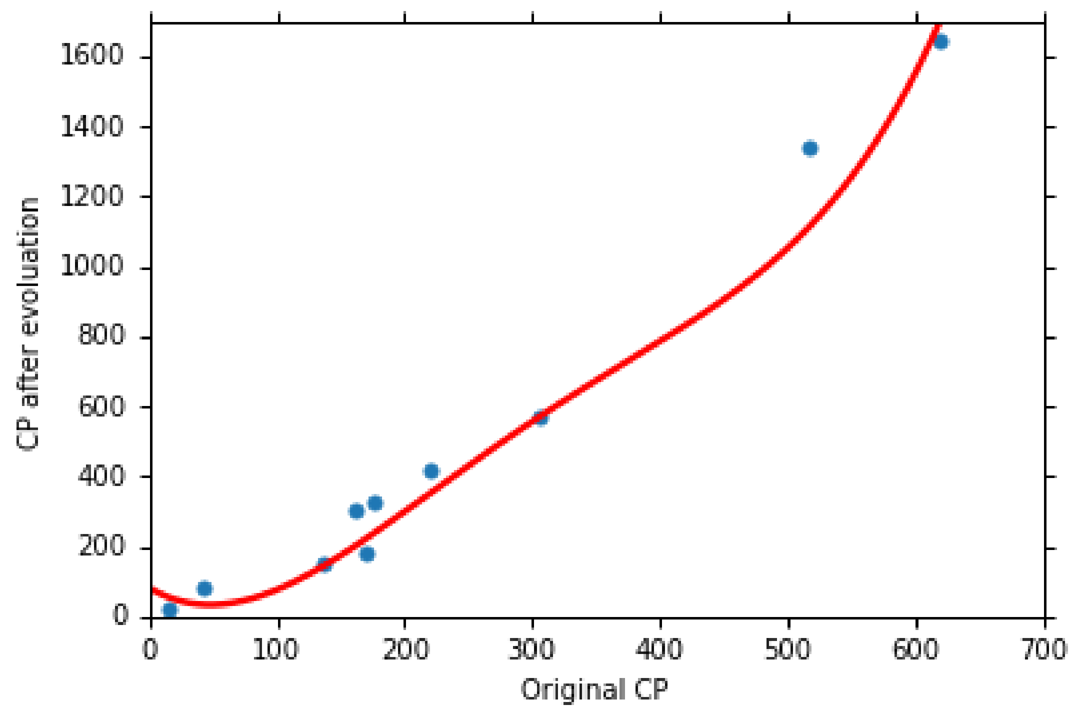
Selecting another Model

$$y = b + w_1 \cdot x_{cp} + w_2 \cdot (x_{cp})^2 + w_3 \cdot (x_{cp})^3 + w_4 \cdot (x_{cp})^4$$



Best Function

Average Error = 14.9



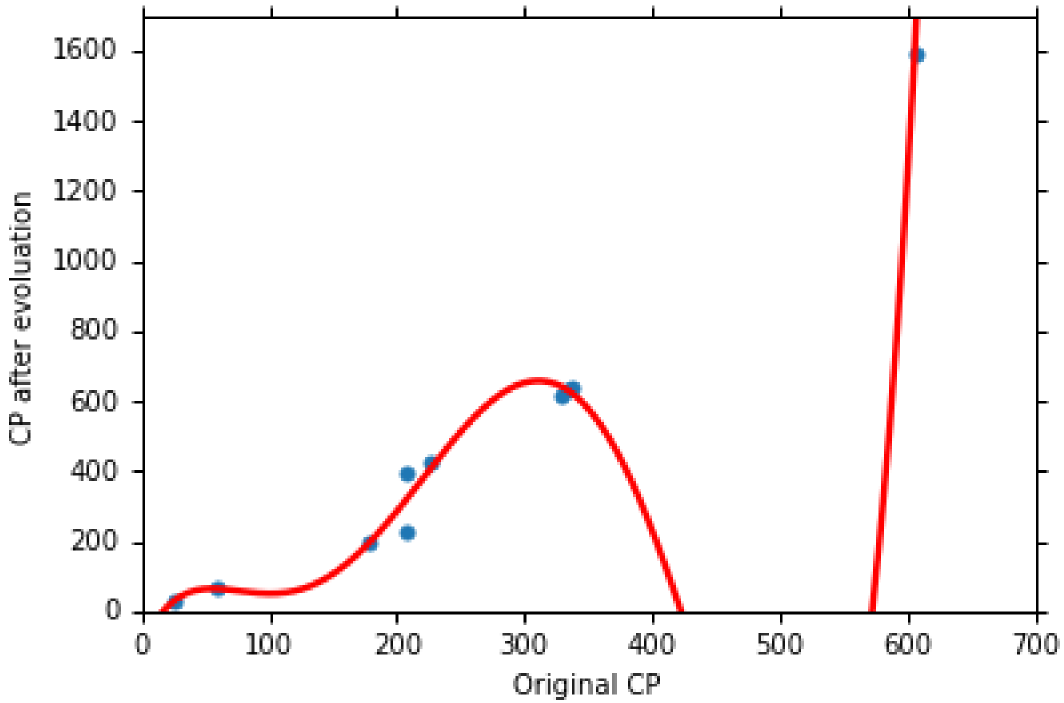
Testing:

Average Error = 28.8

The results become worse ...

Selecting another Model

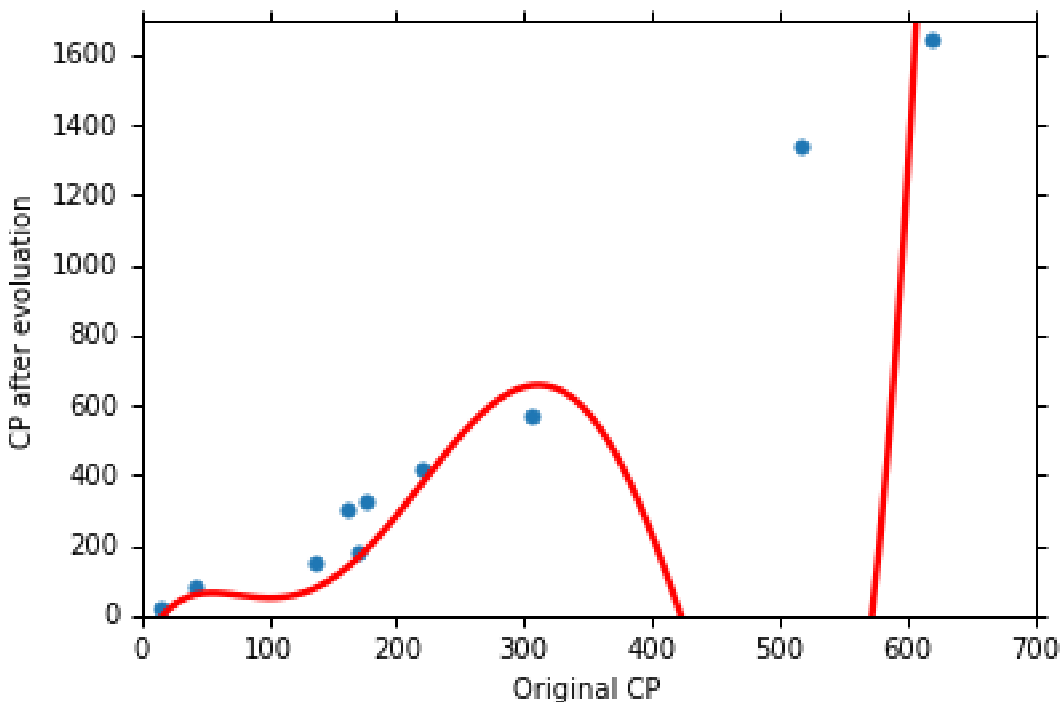
$$y = b + w_1 \cdot x_{cp} + w_2 \cdot (x_{cp})^2 + w_3 \cdot (x_{cp})^3 + w_4 \cdot (x_{cp})^4 + w_5 \cdot (x_{cp})^5$$



Best Function

Average Error = 12.8

Testing:

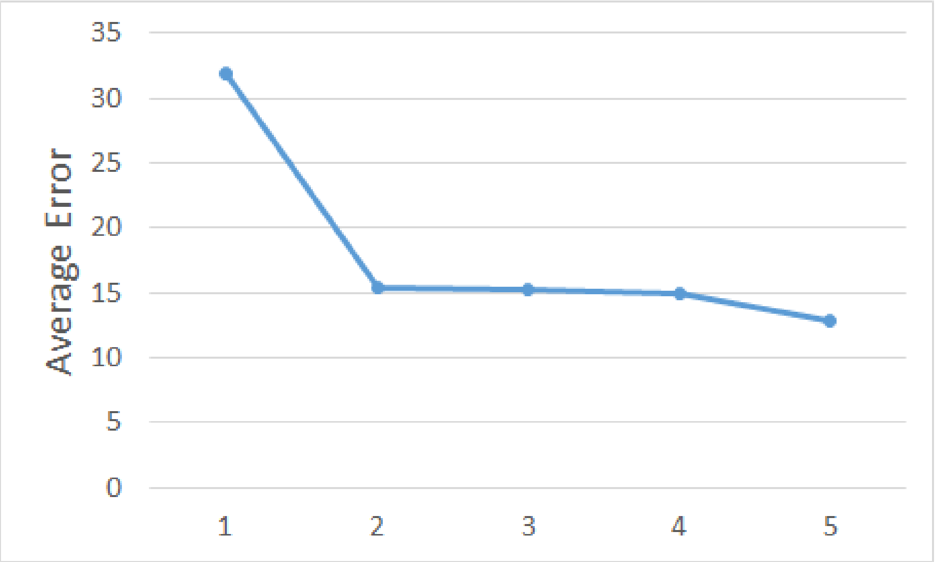


Average Error = 232.1

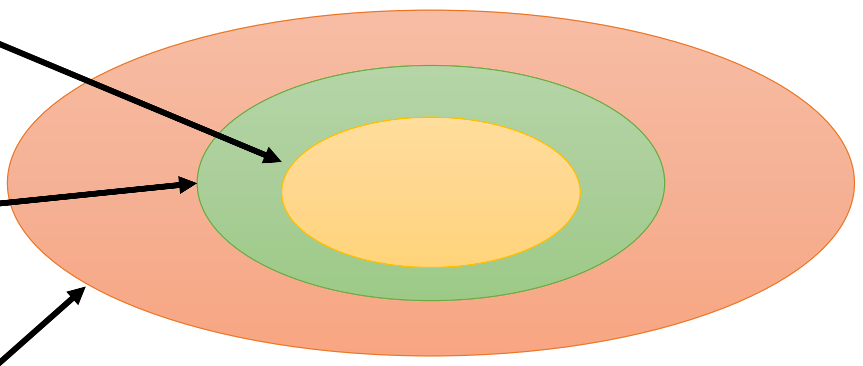
The results are so bad.

Model Selection

Training Data



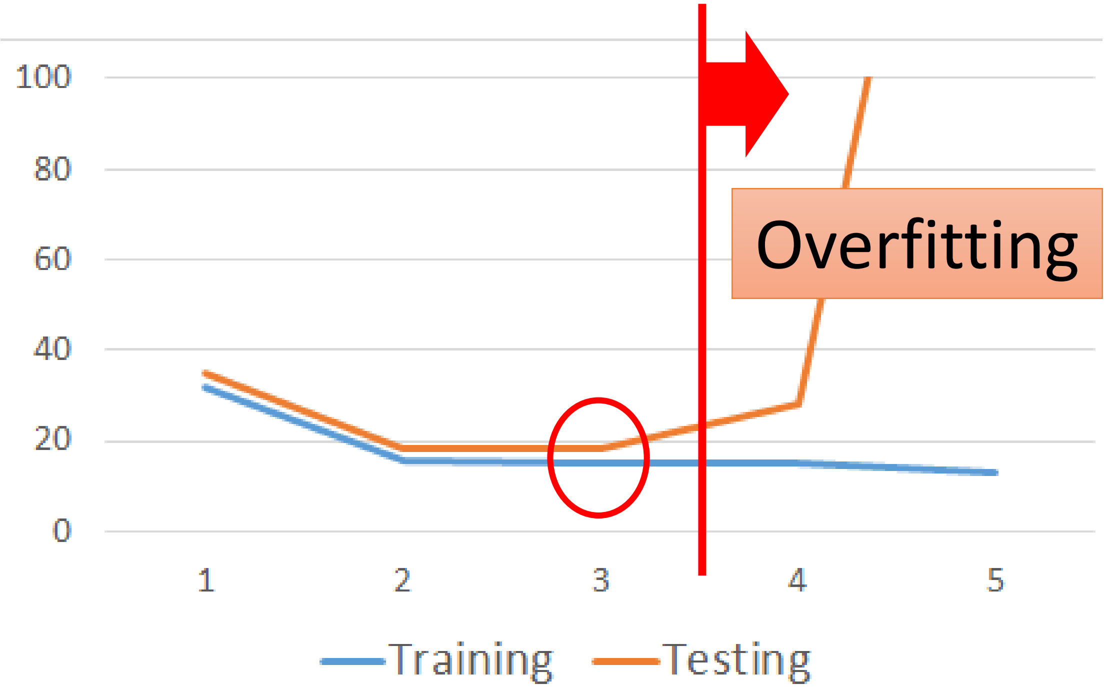
1. $y = b + w \cdot x_{cp}$
2. $y = b + w_1 \cdot x_{cp} + w_2 \cdot (x_{cp})^2$
3. $y = b + w_1 \cdot x_{cp} + w_2 \cdot (x_{cp})^2 + w_3 \cdot (x_{cp})^3$
4. $y = b + w_1 \cdot x_{cp} + w_2 \cdot (x_{cp})^2 + w_3 \cdot (x_{cp})^3 + w_4 \cdot (x_{cp})^4$
5. $y = b + w_1 \cdot x_{cp} + w_2 \cdot (x_{cp})^2 + w_3 \cdot (x_{cp})^3 + w_4 \cdot (x_{cp})^4 + w_5 \cdot (x_{cp})^5$



A more complex model yields lower error on training data.

If we can truly find the best function

Model Selection

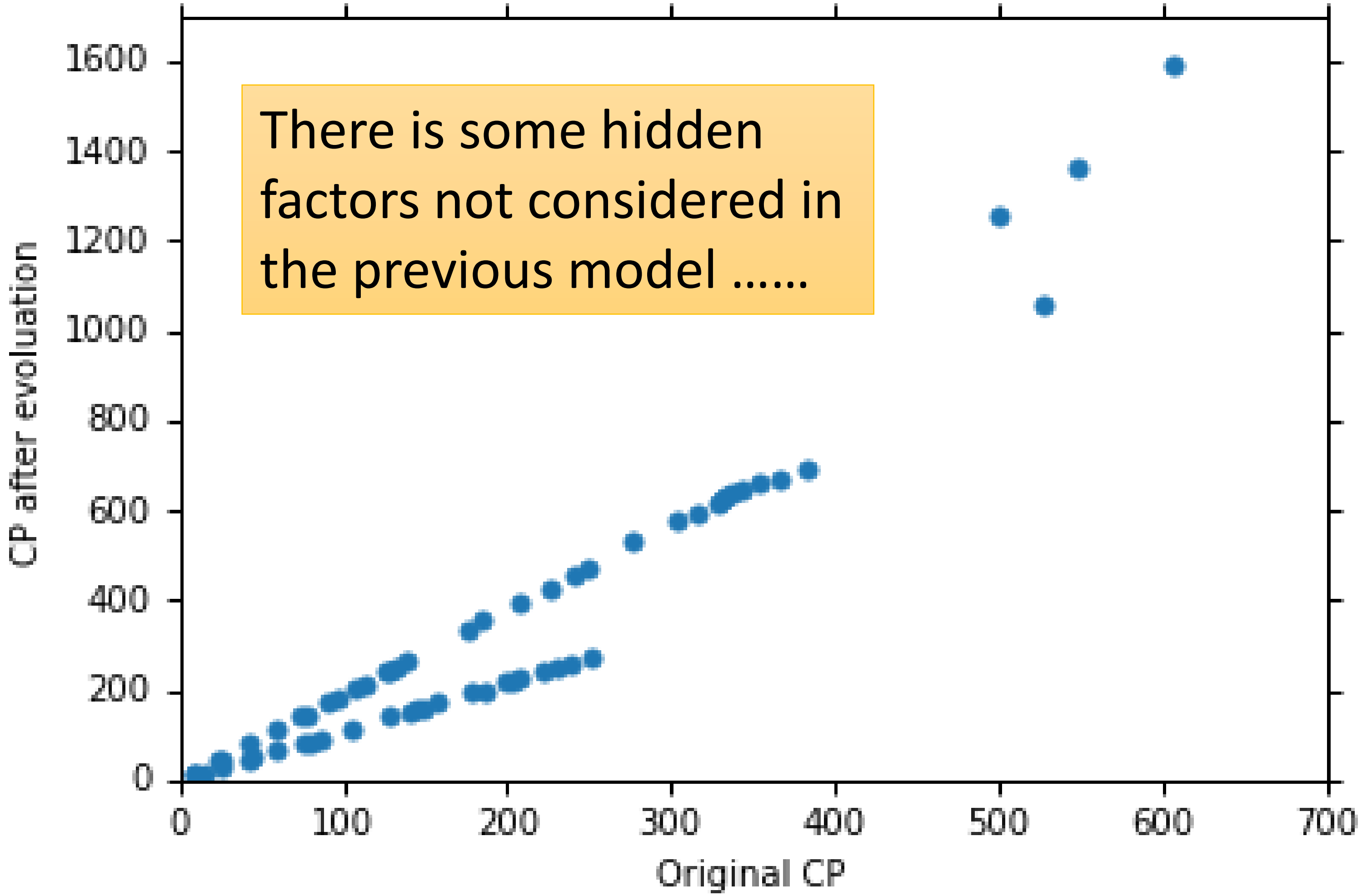


	Training	Testing
1	31.9	35.0
2	15.4	18.4
3	15.3	18.1
4	14.9	28.2
5	12.8	232.1

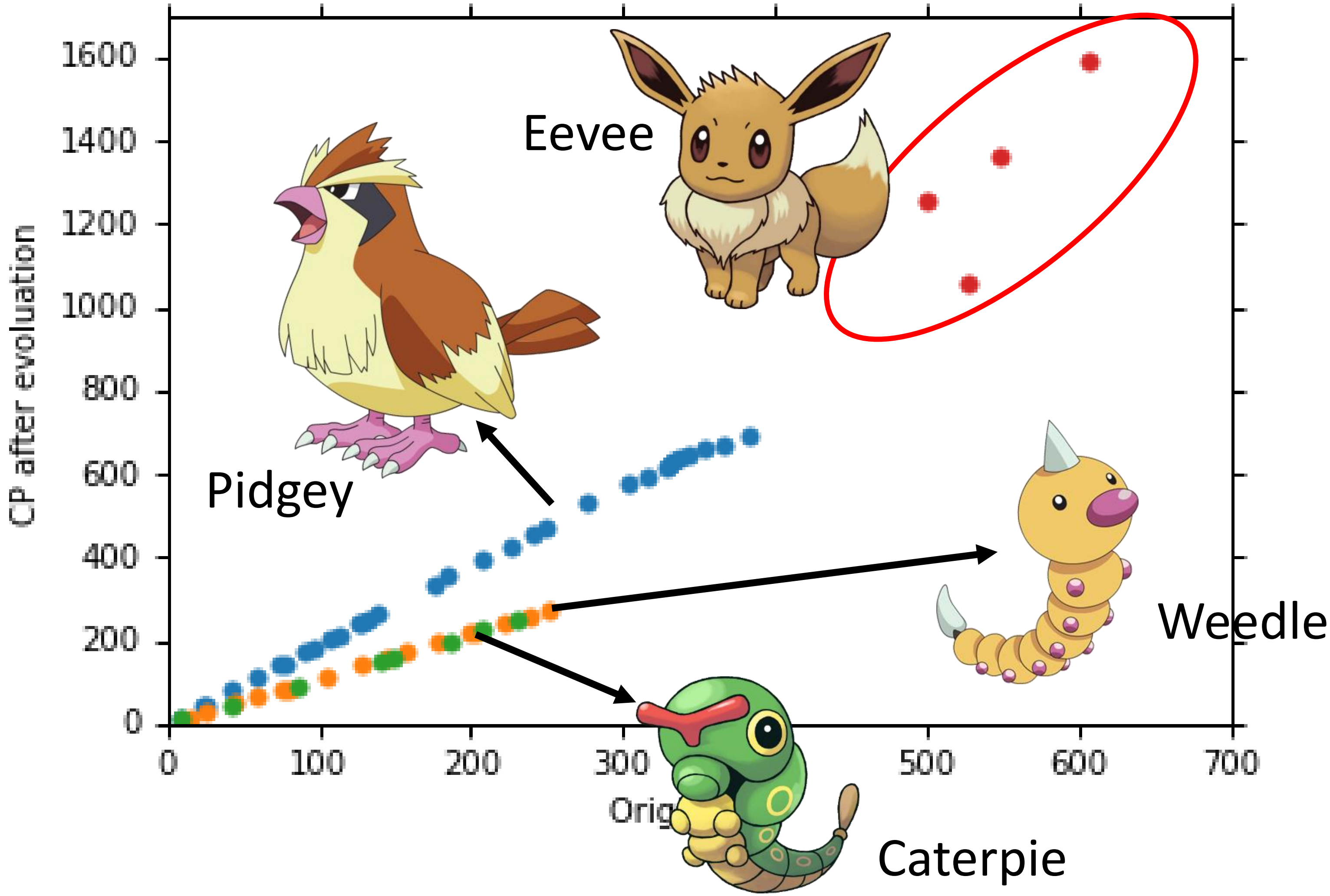
A more complex model does not always lead to better performance on ***testing data***.

This is ***Overfitting***. → Select suitable model

Let’s collect more data



What are the hidden factors?



Back to step 1: Redesign the Model

$$y = b + \sum w_i x_i$$

Linear model?

x_s = species of x

x

If x_s = Pidgey: $y = b_1 + w_1 \cdot x_{cp}$

If x_s = Weedle: $y = b_2 + w_2 \cdot x_{cp}$

If x_s = Caterpie: $y = b_3 + w_3 \cdot x_{cp}$

If x_s = Eevee: $y = b_4 + w_4 \cdot x_{cp}$

y

Back to step 1: Redesign the Model

$$y = b + \sum w_i x_i$$

Linear model?

$y = b_1 \cdot$ 1

$+ w_1 \cdot$ 1 x_{cp}

$+ b_2 \cdot$ 0

$+ w_2 \cdot$ 0

$+ b_3 \cdot$ 0

$+ w_3 \cdot$ 0

$+ b_4 \cdot$ 0

$+ w_4 \cdot$ 0

$\delta(x_s = \text{Pidgey})$

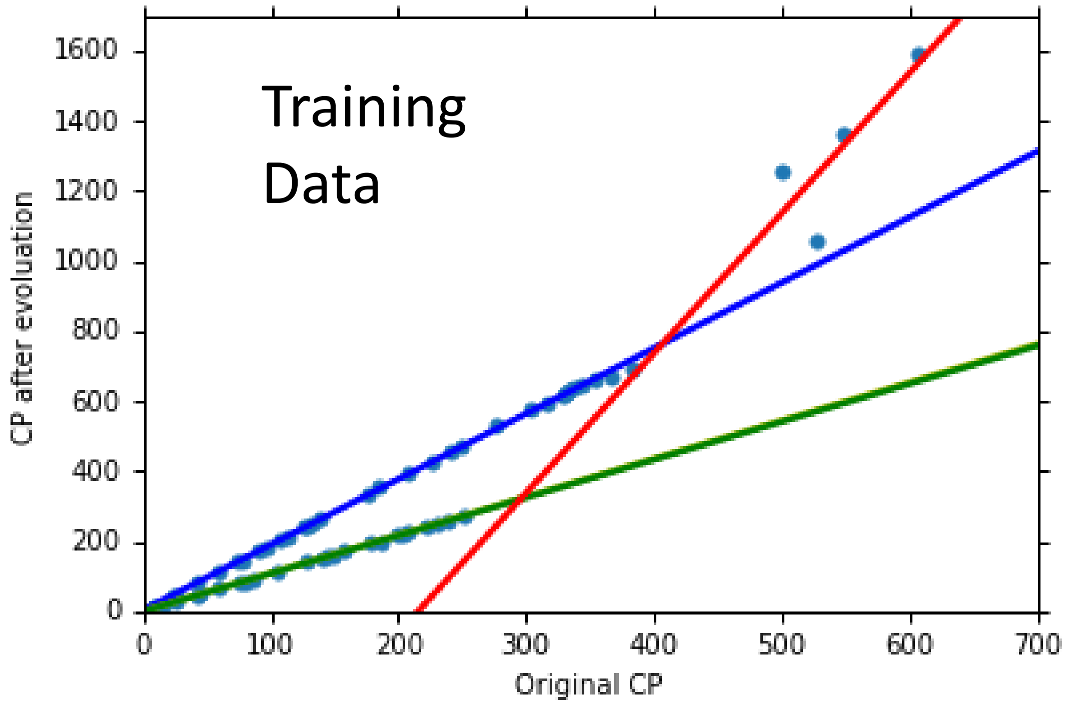
=1 If $x_s = \text{Pidgey}$

=0 otherwise

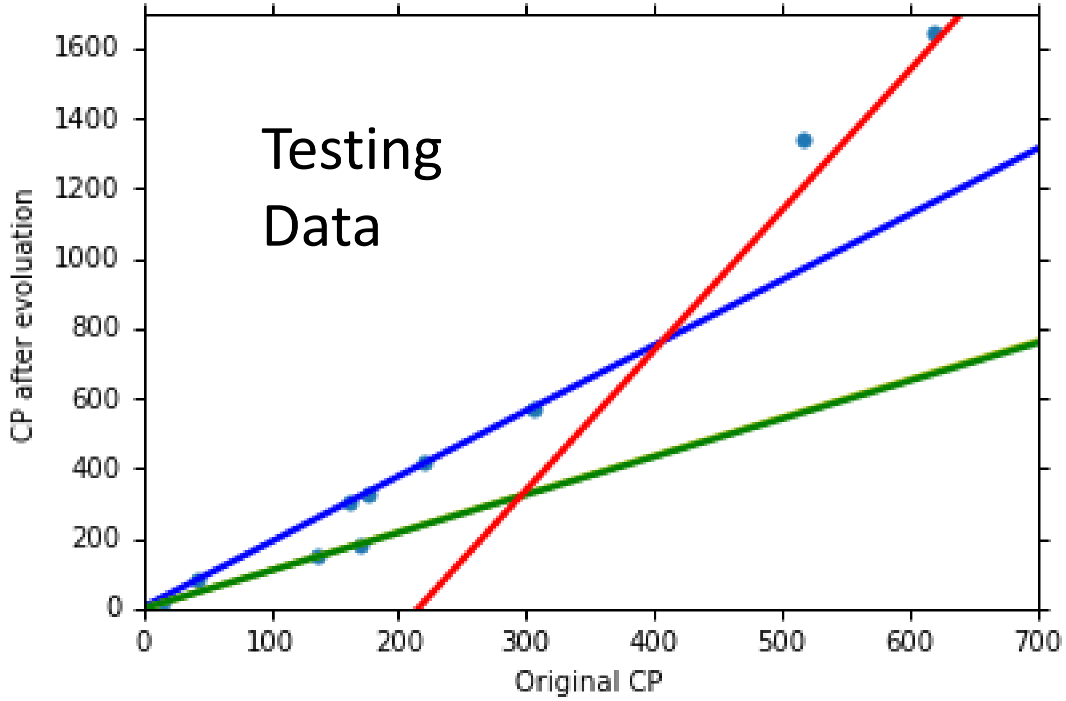
If $x_s = \text{Pidgey}$

$$y = b_1 + w_1 \cdot x_{cp}$$

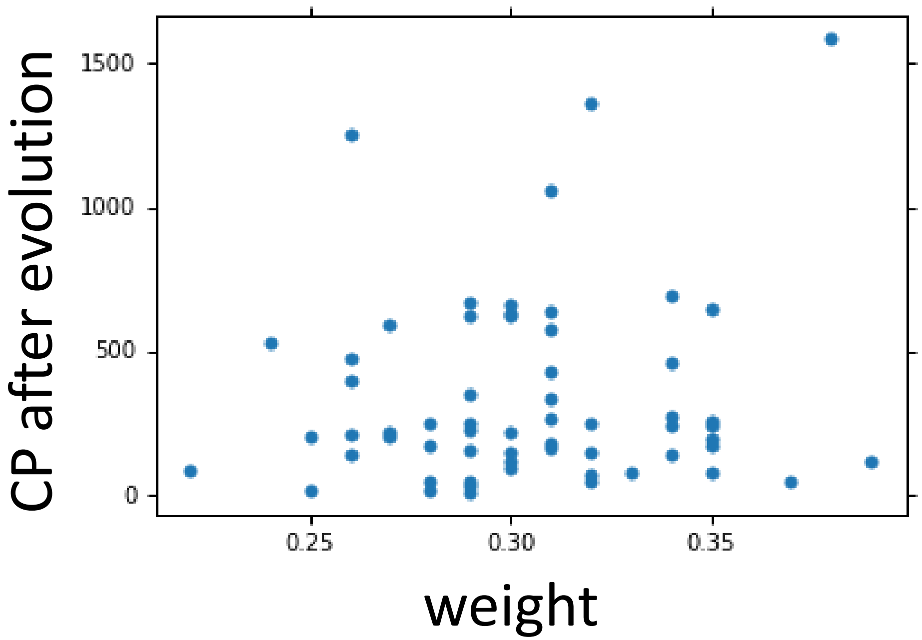
Average error
= 3.8

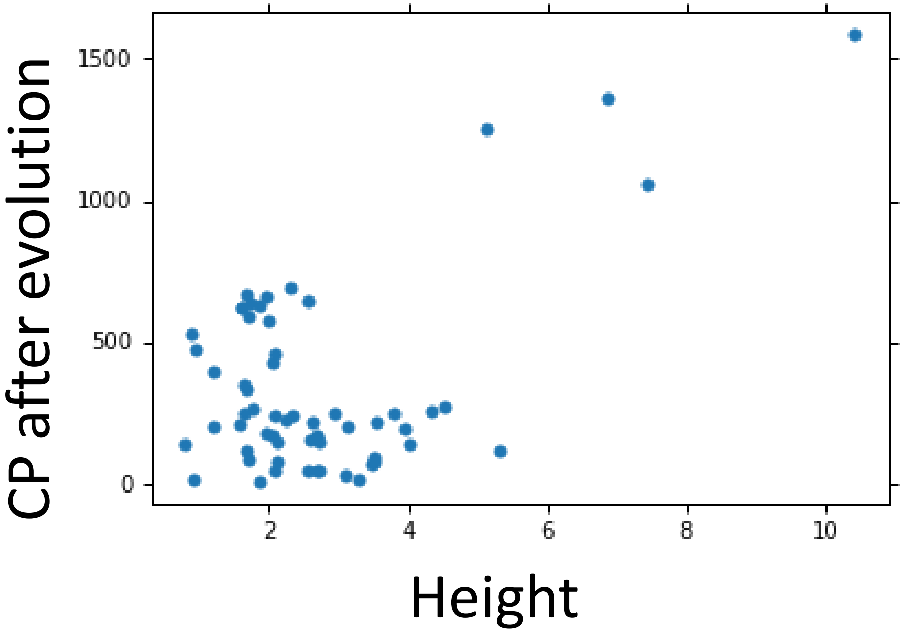


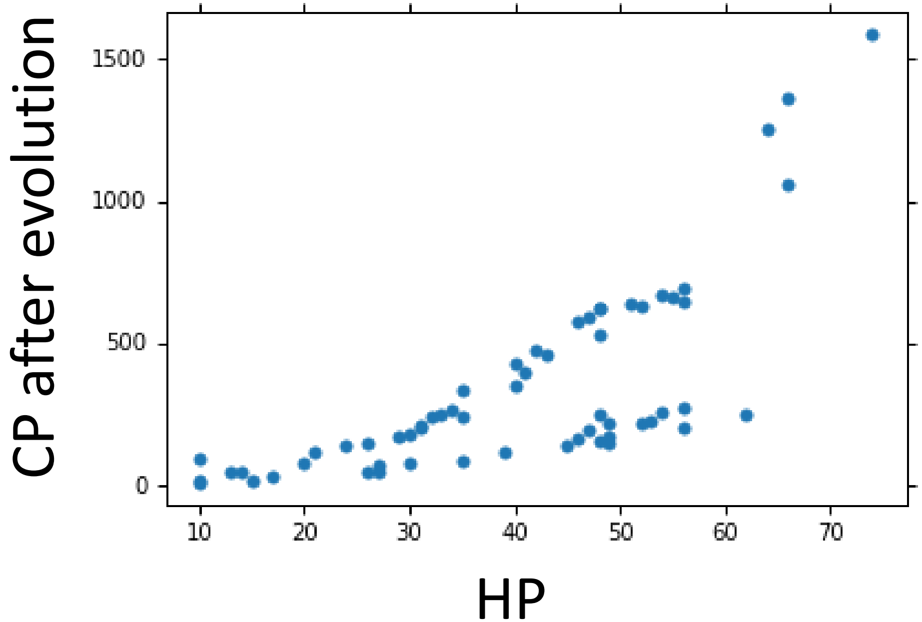
Average error
= 14.3



Are there any other hidden factors?







Back to step 1: Redesign the Model Again

x

If x_s = Pidgey: $y' = b_1 + w_1 \cdot x_{cp} + w_5 \cdot (x_{cp})^2$

If x_s = Weedle: $y' = b_2 + w_2 \cdot x_{cp} + w_6 \cdot (x_{cp})^2$

If x_s = Caterpie: $y' = b_3 + w_3 \cdot x_{cp} + w_7 \cdot (x_{cp})^2$

If x_s = Eevee: $y' = b_4 + w_4 \cdot x_{cp} + w_8 \cdot (x_{cp})^2$

$$y = y' + w_9 \cdot x_{hp} + w_{10} \cdot (x_{hp})^2$$
$$+ w_{11} \cdot x_h + w_{12} \cdot (x_h)^2 + w_{13} \cdot x_w + w_{14} \cdot (x_w)^2$$

y

Training Error = 1.9

Testing Error = 102.3

Overfitting!

Back to step 2: Regularization

$$y = b + \sum w_i x_i$$

$$L = \sum_n \left(\hat{y}^n - \left(b + \sum w_i x_i \right) \right)^2 \quad +\lambda \sum (w_i)^2$$

The functions with smaller w_i are better

- Smaller w_i means … smoother

$$y = b + \sum w_i x_i$$

$$y + \sum w_i \Delta x_i = b + \sum w_i (x_i + \Delta x_i)$$

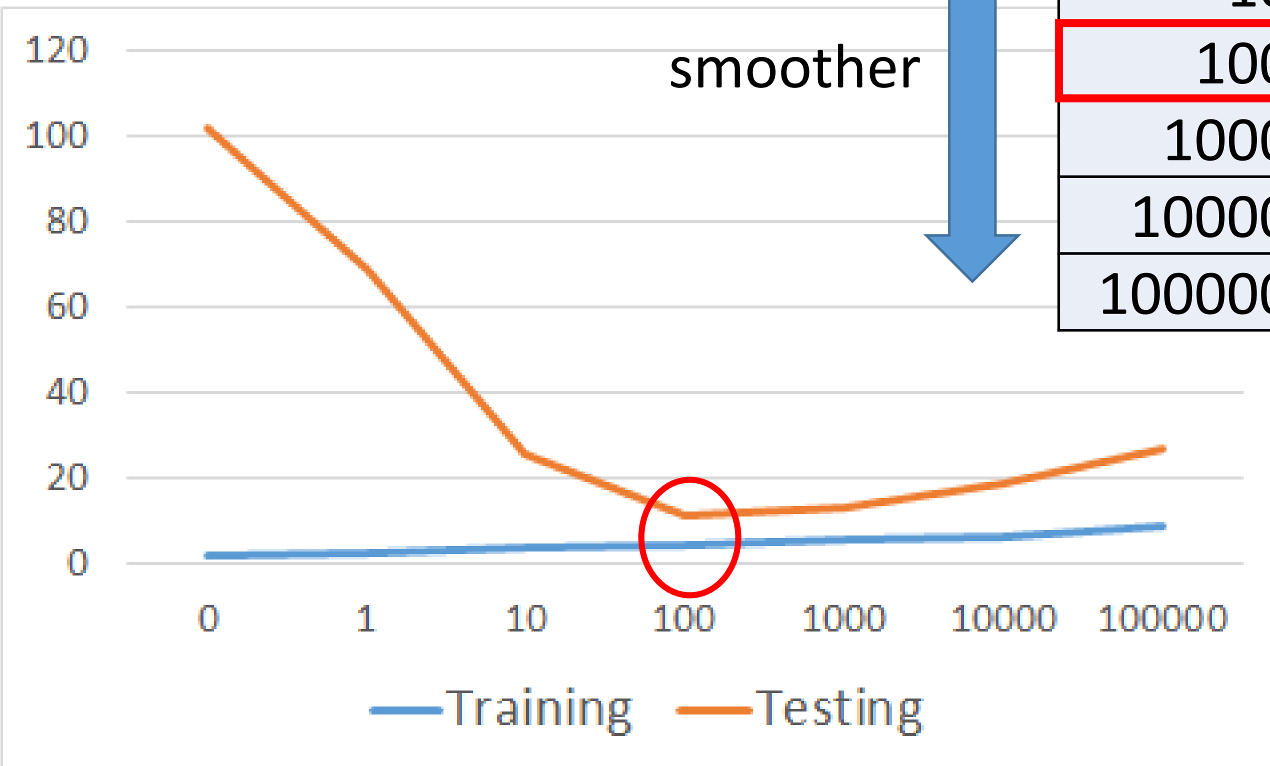
- We believe smoother function is more likely to be correct

Do you have to apply regularization on bias?

Regularization

smoother

λ	Training	Testing
0	1.9	102.3
1	2.3	68.7
10	3.5	25.7
100	4.1	11.1
1000	5.6	12.8
10000	6.3	18.7
100000	8.5	26.8



How smooth?

Select λ obtaining the best model

- Training error: larger λ, considering the training error less
- We prefer smooth function, but don't be too smooth.

Conclusion

- Pokémon: Original CP and species almost decide the CP after evolution
  - There are probably other hidden factors
- Gradient descent
  - More theory and tips in the following lectures
- We finally get average error = 11.1 on the testing data
  - How about new data? Larger error? Lower error?
- Next lecture: Where does the error come from?
  - More theory about overfitting and regularization
  - The concept of validation

Reference

- Bishop: Chapter 1.1

Acknowledgment

- 感謝 鄭凱文 同學發現投影片 上的符號錯誤
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